TOPOLOGICAL m-GROUPS

By

ROBERT LEE RICHARDSON

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INTRODUCTION

In this paper some of the theory of topological m-groups will be developed with some new contributions to the theory of algebraic m-groups. The main result is that any topological m-group can be considered as the coset of an ordinary topological group. While almost no work has been done in the field of topological m-groups, an extensive theory of m-groups has been developed through the years.

In 1928, W Dörnte [4] introduced the concept of an m-group as an extension of a group or 2-group which has as basic operation one that is polyadic instead of dyadic. Previous attempts had been made at this, notably by E. Kasner in an unpublished paper, but his work indicated that he still considered the system to have a basic dyadic operation of which the polyadic operation was merely an extension. Dörnte was the first to publish a paper considering an algebraic system in which the basic operation was polyadic with no underlying dyadic one.

In 1932, D. H. Lehmer [10] introduced the concept of a triplex, which is an abelian 3-group in Dörnte's terminology, apparently without knowledge of Dörnte's work and proceeded to develop a theory of these triplexes.

In 1935, G. A. Miller [13] obtained a result for finite polyadic groups stating that every finite m-group is the coset of an invariant subgroup of some ordinary group. Unfortunately, he makes the tacit assumption that the set of elements in the m-group comes from an ordinary 2-group initially.

A major advancement in the field of m-group theory was achieved by E. L. Post [15] in 1940 when he proved that any m-group (finite or infinite) is the coset of an invariant subgroup of an ordinary group. In addition, he proves in his paper that most of the 2-group concepts, with the notable exception of Sylow's theorem, can be extended to m-groups.

In 1952, H. Tvermoes [25] introduced the concept of an m-semigroup and did a little work on them, but his main interest was again in m-groups.

In 1963-65, F. M. Sioson introduced the concepts of topological m-semigroups and topological m-groups. In papers by him [17], [18], [19], [20], [21], [22], [23], various generalizations of many theorems in 2-semigroups to m-semigroups have been achieved. In a paper with J. D. Monk [24], it is shown that any m-semigroup can be embedded in an ordinary semigroup in such a way that the operations in the m-semigroup reduce to those in the containing semigroup. A representation theorem for m-semigroups then results.

The general theory of such algebraic systems has also been the subject of study in papers by L. M. Gluskin [6], [7] and D. Boccioni [2].

CHAPTER I

SOME PROPERTIES OF ALGEBRAIC m-GROUPS

In this chapter we will develop some elementary ideas concerning algebraic m-groups including a new proof of the Post Coset Theorem [15]. With the exception of [16], very few results on m-groups with idempotents have been published; however, some new theorems concerning m-groups with idempotents are obtained in this chapter.

One of the important problems arising in the study of m-groups is the question of which congruences determine a sub-m-group. In this chapter a set of necessary and sufficient conditions will be given.

Notation 1.0: In the sequel, a sequence of juxtaposed elements $x_i \ldots x_{k+i}$ will be denoted by x_i^{i+k}, and if $x_i = x_{i+1} = \ldots = x_{i+k} = x$ simply as x^{k+1}, k+1 denoting the number of times the element x occurs.

Definition 1.1: An m-semigroup is a set A and a function $f: A^m \longrightarrow A$ such that for all $a_1, \ldots, a_{2m-1} \in A$, $f(f(a_1, \ldots, a_m), a_{m+1}, \ldots, a_{2m-1}) = f(a_1, f(a_2, \ldots, a_{m+1}), a_{m+2}, \ldots, a_{2m-1}) \ldots = f(a_1, a_2, \ldots, a_{m-1}, f(a_m, \ldots, a_{2m-1}))$. Following customary usage, we shall write $[a_1 \ldots a_m] = [a_1^m]$ for $f(a_1, \ldots, a_m)$ and $(A, [\,])$ for (A,f).

Proposition 1.2: For any $k = 1, 2, \ldots,$ $(A, [\,])$ is also a $k(m-1)+1$ semigroup $(A, (\,))$, where $(a_1^{k(m-1)+1}) = [[\ldots[a_1^m]\ldots]a_{(k-1)(m-1)+1}^{k(m-1)+1}]$.

For example, a 2-semigroup A is a $k(2-1)+1 = k+1$ semigroup for any $k = 1, 2, \ldots$. If $k = 2$, then A becomes a 3-semigroup by defining $(a_1a_2a_3) = [[a_1a_2]a_3]$.

Definition 1.3: An m-semigroup $(A, [\,])$ is an m-group iff for each i and for all $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m, b \in A$ there exists uniquely an $x \in A$ such that $[a_1^{i-1}xa_{i+1}^m] = b$.

The following are some examples of m-groups.

Example 1.4: Let R^- be the set of negative real numbers and define $[\,]:(R^-)^3 \longrightarrow R^-$ by $[xyz] = x \cdot y \cdot z$, the ordinary product of x, y, and z. Then $(R^-, [\,])$ forms a 3-group.

Example 1.5: Let $S_1, S_2, \ldots, S_{m-1}$ be any m-1 sets of the same cardinality. Let $F(S_1, \ldots, S_{m-1})$ be the family of all bijective and surjective functions $f: \cup_{i=1}^{m-1} S_i \longrightarrow \cup_{i=1}^{m-1} S_i$ such that $f(S_i) = S_{p(i)}$ where p is any fixed permutation of 1, 2, ..., m-1. Let $[\,]: F^m \longrightarrow F$ be defined as the composition of m functions, i.e. $[f_1 \ldots\ldots f_m](x) = (f_1 \ldots\ldots (f_m(x)))$. $(F, [\,])$ is an m-group since it is clear that $[\,]$ is m-associative and if $x \in S_i$, $f_m(x) \in S_{p(i)}$, $f_{m-1}(f_m(x)) \in S_{p(p(i))}, \ldots\ldots$, $(f_1 \ldots (f_m(x)))$ which is an element of $S_{p^m(i)}$ where $p^m(i)$ is the m^{th} permutation of i. $p^{m-1}(i)$ is i, so the m^{th} permutation of i is p(i). Hence,

$[f_1 \ldots f_m]: S_i \longrightarrow S_{p(i)}$ so that $[f_1 \ldots f_m] \in F$. Unique solvability follows from the fact that the functions are bijective and surjective.

Example 1.6: Let Z be the integers. Then (Z, [ ]) forms an m-group when $[\,]: Z^m \longrightarrow Z$ is defined by $[x_1 \ldots x_m] = x_1 + \ldots + x_m + h$ for any $h \in I$.

Example 1.7: More generally, if (G,·) is an ordinary group and if h is in the center of G, (G,[ ]) will be an m-group if $[\,]: G^m \longrightarrow$ is defined as $[x_1 \ldots x_m] = x_1 \cdot x_2 \cdots x_m \cdot h$.

Example 1.8: Let $\xi, \xi^2, \ldots, \xi^{m-1} = 1$ be any complex $(m-1)^{th}$ roots of unity. Define $[\,]: \mathbb{C}^m \longrightarrow \mathbb{C}$ by $[a_1 \ldots a_m] = a_1 + a_2\xi + \ldots + a_m\xi^{m-1}$. $(\mathbb{C},[\,])$ forms an m-group for it is clear that all conditions except m-associativity are satisfied, and that condition being satisfied is apparent if the following two expansions are studied: $[[a_1^m]a_{m+1}^{2m-1}] = [a_1^m] + a_{m+1}\xi + \ldots + a_{2m-1}\xi^{m-1} = a_1 + a_2\xi + \ldots + a_i\xi^{i-1} + a_{i+1}\xi^i + \ldots + a_m\xi^{m-1} + a_{m+1}\xi + \ldots + a_{m+i+1}\xi^{i+1} + \ldots + a_{2m-1}\xi^{m-1}$. $[a_1^i[a_{i+1}^{i+m}]a_{i+m+1}^{2m-1}] = a_1 + a_2\xi + \ldots + a_i\xi^{i-1} + [a_{i+1}^{i+m}]\xi^i + a_{i+m+1}\xi^{i+1} + \ldots + a_{2m-1}\xi^{m-1} = a_1 + a_2\xi + \ldots + a_i\xi^{i-1} + a_{i+(m-i)}\xi^{i+m-i-1} + a_{i+(m+1-i)}\xi^{i+m+1-i-1} + \ldots + a_{2m-1}\xi^{m-1}$. Equality is apparent by noting that $a_{i+(m+1-i)}\xi^{i+m+1-i-1} = a_{m+1}\xi^m = a_{m+1}\xi^{m-1}\xi = a_{m+1}\xi$.

Example 1.9: Let Z_{odd} be the odd integers under the operation $[x_1\ x_2\ x_3] = x_1 + x_2 + x_3$. Then Z_{odd} is a 3-group.

Example 1.10: Let $V_1, V_2, \ldots, V_{m-1}$ be finite dimensional vector spaces of the same dimension n. Let $L(V_1, \ldots, V_{m-1})$ be the set of all (m-1)-tuples of non-singular linear transformations $(A_1, A_2, \ldots, A_{m-1})$ where $A_i : V_i \longrightarrow V_{p(i)}$ for the permutation $p = (12\ldots m-1)$. L is an m-group under the operation $[(A_1^1, \ldots, A_{m-1}^1)(A_1^2, \ldots, A_{m-1}^2)\ldots(A_1^m, \ldots, A_{m-1}^m)] = (A_1^1 A_2^2 \ldots A_1^m, A_2^1 A_3^2 \ldots A_{m-1}^{m-2} A_1^{m-1} A_2^m, \ldots, A_{m-1}^1 A_1^2 A_2^3 \ldots A_{m-1}^m)$. Associativity is clear, and unique solvability follows from the condition that the linear transformations be non-singular.

Definition 1.11: An (m-1)-tuple $(e_1, e_2, \ldots, e_{m-1})$ of elements $e_i \in A$ is a left (right) m-adic identity iff for all $x \in A$, $[e_1^{m-1} x] = x$ ($[x e_1^{m-1}] = x$). When $(e_1, e_2, \ldots, e_{m-1})$ is both a left and right m-adic identity it is simply called an m-adic identity.

Proposition 1.12: For any $a, e_1, e_2, \ldots, e_{m-1} \in A$, if $[e_1^{m-1} a] = a$ ($[a e_1^{m-1}] = a$), then $(e_1, e_2, \ldots, e_{m-1})$ is a left (right) m-adic identity.

Proof. Let $x \in A$ be arbitrary and $[e_1^{m-1} a] = a$. Choose $a_2, \ldots, a_m \in A$ such that $[a a_2^m] = x$. Then $x = [a a_2^m] = [[e_1^{m-1} a] a_2^m] = [e_1^{m-1} x]$. The other part is proved in a similar fashion.

Proposition 1.13: Every left m-adic identity in an m-group is also a right m-adic identity and conversely.

Proof. Let $(e_1, e_2, \ldots, e_{m-1})$ be a left m-adic identity. Note that $[a e_1^{m-2} [e_{m-1} e_1^{m-1}]] = [a [e_1^{m-1} e_1] e_2^{m-1}]$

$= [ae_1^{m-1}]$. From the definition of an m-group it then follows that $[e_{m-1}\, e_1^{m-1}] = e_{m-1}$ which by Proposition 1.12 implies that $(e_1, e_2, \ldots, e_{m-1})$ is also a right m-adic identity.

Proposition 1.14: If $(e_1, e_2, \ldots, e_{m-1})$ is an m-adic identity, then for each i, $(e_i, \ldots, e_{m-1}, e_1, \ldots, e_{i-1})$ is also an m-adic identity in the m-group.

Proof. If $(e_1, e_2, \ldots, e_{m-1})$ is an m-adic identity, then $[e_1\, e_2 \ldots e_{m-1}\, e_1] = e_1$ and hence by Propositions 1.12 and 1.13 it is also true that $(e_2, \ldots, e_{m-1}, e_1)$ is an m-adic identity. By a repetition of this argument, the result follows.

Definition 1.15: An element x in the m-group will be called an idempotent if and only if $[x^m] = x$.

Proposition 1.16: If x is an idempotent in an m-group A, then (x, x, ..., x) (m-1) times is both a left and right m-adic identity.

Proof. (x, x, ..., x) (m-1) times is a left and right identity on x; hence, by Proposition 1.12, for all $z \in A$. By Proposition 1.13, (x, ..., x) (m-1) times is both a left and right m-adic identity.

Definition 1.17: The inverse of an (m-2)-tuple $(x_1, x_2, \ldots, x_{m-2})$ of elements from an m-group is the unique element x_{m-1} also denoted by $(x_1, x_2, \ldots, x_{m-2})^{-1}$ such that $(x_1, x_2, \ldots, x_{m-1})$ is an m-adic identity. We note that such an element always exists by the definition of an m-group and Proposition 1.12.

Definition 1.18: Let $S_1, \ldots, S_{m-1}$ be any $(m-1)$ sets. An m-adic function on $S_1, \ldots, S_{m-1}$ is a function

$$f: \bigcup_{i=1}^{m-1} S_i \longrightarrow \bigcup_{i=1}^{m-1} S_i$$

such that $f(S_i) \subseteq S_{\sigma(i)}$ where $\sigma = (12\ldots m-1)$.

Proposition 1.19: The family of all surjective and bijective m-adic functions on sets $S_1, \ldots, S_{m-1}$ of the same cardinality forms an m-group.

Proof. See Example 1.5.

Definition 1.20: Two k-tuples ($k < m$) of elements from an m-group A are equivalent, i.e., $(a_{i+1}, \ldots, a_{i+k}) \equiv (b_{i+1}, \ldots, b_{i+k})$, iff for all $x_1, \ldots, x_i, x_{i+k+1}, \ldots, x_m \in A$ $(0 \leq i, i+k \leq m)$,

$$[x_1^i a_{i+1}^{i+k} x_{i+k+1}^m] = [x_1^i b_{i+1}^{i+k} x_{i+k+1}^m].$$

Note that by the above definition $(a_1, \ldots, a_m) \equiv (b_1, \ldots, b_m)$ if and only if $[a_1^m] = [b_1^m]$.

Proposition 1.21: $(a_{i+1}, \ldots, a_{i+k}) \equiv (b_{i+1}, \ldots, b_{i+k})$ iff there exists $c_1, \ldots, c_i, c_{i+k+1}, \ldots, c_m \in A$ such that

$$[c_1^i a_{i+1}^{i+k} c_{i+k+1}^m] = [c_1^i b_{i+1}^{i+k} c_{i+k+1}^m].$$

Proof. Let $d_{i+1}, \ldots, d_{m-1}, d_2, \ldots, d_{i+k} \in A$ such that $(d_{i+1}, \ldots, d_{m-1}, c_1, \ldots, c_i)$ and $(c_{i+k+1}, \ldots, c_m, d_2, \ldots, d_{i+k})$ are m-adic identities of A. Then for each

$x_1, \ldots, x_i, x_{i+k+1}, \ldots, x_m \in A$,

$$[x_1^i a_{i+1}^{i+k} x_{i+k+1}^m]$$

$$= [x_1^i [d_{i+1}^{m-1} c_1^i a_{i+1}] a_{i+2}^{i+k-1} [a_{i+k} c_{i+k+1}^m d_2^{i+k}] x_{i+k+1}^m]$$

$$= [[x_1^i d_{i+1}^{m-1} [c_1^i a_{i+1}^{i+k} c_{i+k+1}^m [[d_2^{i+k} x_{i+k+1}^m]$$

$$= [[x_1^i d_{i+1}^{m-1} [c_i^i b_{i+1}^{i+k} c_{i+k+1}^m]] d_2^{i+k} x_{i+k+1}^m]$$

$$= [x_1^i [d_{i+1}^{m-1} c_1^i b_{i+1}] b_{i+2}^{i+k-1} [b_{i+k} c_{i+k+1}^m d_2^{i+k}] x_{i+k+1}^m]$$

$$= [x_1^i b_{i+1}^{i+k} x_{i+k+1}^m].$$

Proposition 1.22: $\equiv$ is an equivalence relation.

Proof. That $\equiv$ is reflexive and symmetric is clear. Suppose $(a_{i+1}, \ldots, a_{i+k}) \equiv (b_{i+1}, \ldots, b_{i+k})$ and $(b_{i+1}, \ldots, b_{i+k}) \equiv (c_{i+1}, \ldots, c_{i+k})$. By Definition 1.20, for any $x_1, \ldots, x_i, x_{i+k+1}, \ldots, x_m \in A$, $[x_1^i a_{i+1}^{i+k} x_{i+k+1}^m] = [x_1^i b_{i+1}^{i+k} x_{i+k+1}^m] = [x_1^i c_{i+1}^{i+k} x_{i+k+1}^m]$. Hence, $(a_{i+1}, \ldots, a_{i+k}) \equiv (c_{i+1}, \ldots, c_{i+k})$.

Let $S_i = A^i/\equiv$. Note $S_1 = A$.

Theorem 1.23: Any m-group is isomorphic to an m-group of bijective and surjective m-adic functions on disjoint sets $S_1, \ldots, S_{m-1}$.

Proof. Let $F(S_1, \ldots, S_{m-1})$ be the family of all surjective and bijective m-adic functions on $S_i = A^i/\equiv$, $i = 1, \ldots, m-1$. For each $a \in A$, define $L_a: \bigcup_{i=1}^{m-1} S_i \longrightarrow \bigcup_{i=1}^{m-1} S_i$ such that $L_a((x_1, \ldots, x_i)/\equiv) = (a, x_1, \ldots, x_i)/\equiv$, $i = 1, \ldots, m-2$, and $L_a((x_1, \ldots, x_{m-1})/\equiv) = [ax_1^{m-1}]$.

This is well defined since if $(x_1, \ldots, x_i) \equiv (y_1, \ldots, y_i)$, then also $(a, x_1, \ldots, x_i) \equiv (a, y_1, \ldots, y_i)$.

Suppose $L_a((x_1, \ldots, x_i)/\equiv) = L_a((y_1, \ldots, y_i)/\equiv)$ so that

$$(a, x_1, \ldots, x_i) \equiv (a, y_1, \ldots, y_i)$$

and hence $[ax_1^i a_{i+1}^{m-1}] = [ay_1^i a_{i+1}^{m-1}]$ for some $a_{i+1}, \ldots, a_{m-1} \in A$. Thus, by Proposition 1.21, $(x_1, \ldots, x_i)/\equiv = (y_1, \ldots, y_i)/\equiv$; that is to say, L_a is bijective.

Let $(y_1, \ldots, y_{i+1})/\equiv \in S_{i+1}$. Then for $a_{i+2}, \ldots, a_m \in A$, there exists uniquely (by definition of an m-group) an $x \in A$ such that

$$[ay_2^i x a_{i+2}^m] = [y_1^{i+1} a_{i+2}^m].$$

This means that $L_a((y_2, \ldots, y_i, x)/\equiv = (y_1, \ldots, y_{i+1})/\equiv$. Thus L_a is also surjective.

Define $f: A \longrightarrow F(S_1, \ldots, S_{m-1})$ such that $f(a) = L_a$.

Note that

$$([a_1^m], x_1, \ldots, x_i) \equiv (a_1, [a_2^m x_1], \ldots, x_i)$$

$$([a_2^m x_1], \ldots, x_i) \equiv (a_2, [a_3^m x_1^2], \ldots, x_i)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$([a_i^m x_1^{i-1}], x_i) \equiv (a_i, [a_{i+1}^m x_1^i]),$$

$$([a_{i+1}^m x_1^i]) \equiv (a_{i+1}, \ldots, a_m, x_1, \ldots, x_i).$$

Thus,

$$L_{[a_1^m]}((x_1, \ldots, x_i)/\equiv) = ([a_1^m], x_1, \ldots, x_i)/\equiv$$

$$= L_{a_1}([a_2^m x_1], x_2, \ldots, x_i)/\equiv) = \ldots$$

$$= L_{a_1} L_{a_2} \ldots L_{a_m}((x_1, \ldots, x_i)/\blacksquare).$$

Whence $f([a_1^m]) = L_{[a_1^m]} = L_{a_1} L_{a_2} \ldots L_{a_m} = f(a_1)f(a_2)\ldots f(a_m)$.

f is also one-to-one. For, if $L_a = f(a) = f(b) = L_b$, then $L_a((x_1, \ldots, x_{m-1})/\blacksquare) = L_b((x_1, \ldots, x_{m-1})/\blacksquare)$ and therefore $[ax_1^{m-1}] = [bx_1^{m-1}]$ or $a = b$ by Definition 1.3.

Next we shall prove the Post Coset Theorem. Other proofs may be found in Bruck [3] and Post [15]. The analogue of this theorem for m-semigroups has been proved by Los [11] for $m = 3$ and later for arbitrary m by Sioson and Monk [24].

Theorem 1.24: (Post Coset Theorem). Let $(A,[\,])$ be an m-group. Then there exists a group $(G,\cdot)$ and normal subgroup N of G such that G/N is cyclic and $A = xN$ with $[a_1^m] = a_1 \cdot a_2 \cdots a_n$ for all $a_1, a_2, \ldots, a_m \in A$. In fact, $G/N = \{A, \ldots, A^{m-1}\}$, $N = A^{m-1}$ and the order of G/N divides m-1.

Proof. By the representation theorem for m-groups, every m-group is isomorphic to an m-subgroup of the m-group of surjective and bijective m-adic functions on $S_1, \ldots, S_{m-1}$. Let G be the group generated by A.

Note that for a fixed $a \in A$ and any element $b = a_1 \ldots a_{m-1} \in A^{m-1}$, there exist uniquely $x, y \in A$ such that

$$[a_1^{m-1}x] = a_1 \ldots a_{m-1}x = a$$

$$[ya_1^{m-1}] = ya_1 \ldots a_{m-1} = a.$$

Thus every element of A^{m-1} can be expressed as ax^{-1} or $y^{-1}a$ for any fixed $a \in A$ and $x, y \in A$. Thus, if $b_1 = y_1^{-1}a$,

$b_2 = y_2^{-1}a$ are any two elements of A^{m-1}, then $b_1 b_2^{-1} = (y_1^{-1}a)(y_2^{-1}a)^{-1} = y_1^{-1}aa^{-1}y_2 = y_1^{-1}y_2$ is also an element of A^{m-1}. This means A^{m-1} is a subgroup of G. Note also that for each $a \in A$, $aA^{m-1} = A = A^{m-1}a$. Thus $a^{-1}A^{m-1}a = A^{m-1} = aA^{m-1}a^{-1}$ for each $a \in A$. Since A generates G, then any $g \in G$ may be written as $g = a_1^{i_1}\ldots a_n^{i_n}$ for $a_i \in A$ and $i_k = -1$ or 1. Then $g^{-1}A^{m-1}g = A^{m-1}$ and A^{m-1} is a normal subgroup of G. From $aA^{m-1} = A$ we obtain $a^{-1}aA^{m-1} = a^{-1}A = A^{m-1}$. Thus $a^{-1}A^{m-1} = (a^{-1}A)A^{m-2} = A^{m-2}$. Similarly, for $a_1, a_2 \in A$, $a_1^{-1}a_2^{-1}A^{m-1} = a_1^{-1}A^{m-2} = A^{m-3}$, and so on. Thus

$$G A^{m-1} = A \cup A^2 \cup A^3 \cup \ldots \cup A^{m-1}.$$

Some of the Ai's may be equal. In any case, the order of G/A^{m-1} is a divisor of m-1.

Definition 1.25: G will be called the containing group of A and A^{m-1} the associated group of A.

If the order of G/A is exactly m-1, we shall say that G is a covering group of A. E. L. Post [15] has shown that this can always be achieved for any m-group by considering a free group generated by the elements of the m-group.

The following theorem due to Sioson [20] will prove useful in the sequel.

Theorem 1.26: The following conditions for an m-semigroup are equivalent:

(1) A is an m-group;

(2) For all $i = 1, \ldots, m$, for $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m, b \in A$, there exists an $x \in A$ such that $[a_1^{i-1}xa_{i+1}^m] = b$, i.e., $[a_1^{i-1}Aa_{i+1}^m] = A$;

(3) For some i between 1 and m, for $a_1, \ldots, a_m, b \in A$, there exists an $x \in A$ such that $[a_1^{i-1}xa_{i+1}^m] = b$, i.e. $[a_1^{i-1}Aa_{i+1}^m] = A$.

(4) For each $a_1, a_2, \ldots, a_{m-1} \in A$, $[a_1^{m-1}A] = A = [Aa_1^{m-1}]$;

(5) For each $a \in A$, $[aA^{m-1}] = A = [A^{m-1}a]$;

(6) For all $a_1, \ldots, a_{m-2} \in A$, there exists $a_{m-1} \in A$ such that $(a_1, \ldots, a_{m-1})$ is an m-adic identity.

Proof. (1) implies (2) implies (3) implies (4) implies (5) are obvious. (2) implies (1) by definition of m-group; for, if $x, y \in A$ such that $[a_1^{i-1}xa_{i+1}^m] = [a_1^{i-1}ya_{i+1}^m]$, then for some elements $b_i, \ldots, b_{m-1}, c_2, \ldots, c_i \in A$, $(b_i, \ldots, b_{m-1}, a_1, \ldots, a_{i-1})$, $(a_{i+1}, \ldots, a_m, c_2, \ldots, c_i)$ are m-adic identities and hence $x = y$.

(5) implies (2). Let (5) hold and $a_1, a_2, \ldots, a_m \in A$. Noting $A^m = A$, then

$$[a_1^{i-1}Aa_{i+1}^m] = [a_1^{i-1}A^{2m-1}a_{i+1}^m]$$

$$= [a_1^{i-2}[a_{i-1}A^{m-1}]A[A^{m-1}a_{i+1}]a_{i+2}^m]$$

$$= [a_1^{i-2}A^3a_{i+2}^m] = \ldots = A^m = A.$$

(2) implies (6) as we have already seen.

(6) implies (4). We shall show the existence of an x such that $[xa_2^m] = b$ for any $a_2, \ldots, a_m, b \in A$. Since

$[A^m] = A$, then we may write the above equation as

$[[\ldots[[xb_1^1b_2^1\ldots b_{m-1}^1]b_1^2b_2^2\ldots b_{m-1}^2]\ldots]b_1^{m-2}b_2^{m-2}\ldots b_{m-1}^{m-2}] = b.$

By applying (6), then

$x = [b[b_1^{m-2},b_2^{m-2},\ldots b_{m-1}^{m-2}]^{-1}\ldots[b_1^2,b_2^2,\ldots b_{m-1}^2]^{-1}[b_1^1,b_2^1,\ldots b_{m-1}^1]^{-1}].$

Definition 1.27: A subset S of an m-group (A, [ ]) is called a sub-m-group iff S is closed under the same operation [ ] in A and for each $x_1, x_2, \ldots, x_{m-2} \in S$, $(x_1, x_2, \ldots, x_{m-2})^{-1} \in S$.

Proposition 1.28: If S is a sub-m-group of an m-group A, then $H = S^{m-1}$ is a subgroup of the containing group, in fact, of the associated group of A.

Proof. Let A be an m-group with containing group G. As before, we may assume $A \subset G = A \cup A^2 \cup \ldots \cup A^{m-1}$. Let S be a sub-m-group and $H = S^{m-1} \subset G$. Since S is a sub-m-group of A, for each $x_1, x_2, \ldots, x_{m-2} \in S$, there exists $(x_1, x_2, \ldots, x_{m-2})^{-1} = x_{m-1} \in S$ such that $(x_1, x_2, \ldots, x_{m-1})$ is an m-adic identity. Hence $x_1x_2 \ldots x_{m-1}$ as an element of G is the identity and is also an element of H. Next, let $x_1, x_2, \ldots, x_{m-1}$ be m-1 elements of S. Then for any $y_1, y_2, \ldots, y_{m-2} \in S$ there is a $y_{m-1} \in S$ such that $([x_1^{m-1}y_1], y_2, \ldots, y_{m-1})$ is an m-adic identity which implies then that $(x_1x_2 \ldots x_{m-1})(y_1y_2 \ldots y_{m-1})$ is the identity of H. Hence $y_1y_2 \ldots y_{m-1}$ is the inverse of $x_1x_2 \ldots x_{m-1}$ an arbitrary element of H. Also note that $H \cdot H = S^{m-1}\,S^{m-1} = S^m\,S^{m-2} = S^{m-1} = H$. Thus H is closed

both under the binary operation in G and inversion and hence is a subgroup of G and hence of A^{m-1}, the associated group of A.

Definition 1.29: A sub-m-group S of A is called invariant iff $[a_1^i\ S\ a_{i+1}^{m-1}] = S$ for each m-adic identity $(a_1, a_2, \ldots, a_{m-1})$ of A and each $i = 1, 2, \ldots, m-2$ and also $[a_1^{m-1}[S\ a_m^{2m-2}]] = S$ for each $a_1, a_2, \ldots, a_{2m-2}$ such that $([a_1^m], a_{m+1}, \ldots, a_{2m-2})$ is an m-adic identity.

Definition 1.30: Let S be a sub-m-group of A. If the associated subgroup of S in the containing group G of A is invariant in G, then S is called semi-invariant.

Proposition 1.31: Every invariant sub-m-group is semi-invariant.

Proof. Let S be an invariant sub-m-group of A with the associated subgroup $H = S^{m-1}$ in the containing group $G = A \cup A^2 \cup \ldots \cup A^{m-1}$ of A. Let $x \in G$ so that $x = a_1 a_2 \ldots a_i$ for some $i \leq m-1$ and $a_1, a_2, \ldots, a_i \in A$. If $i < m-1$, then there are $a_{i+1}, \ldots, a_{m-1} \in A$ such that $(a_1, a_2, \ldots, a_{m-1})$ is an m-adic identity. Thus $x^{-1} = a_{i+1} \ldots a_{m-1}$ is the inverse of x in G. S is invariant, so $a_1^i\ S\ a_{i+1}^{m-1} = S$. Since the group operation gives the m-group operation, then $H = S^{m-1} = (a_1^i\ S\ a_{i+1}^{m-1})^{m-1} =$

$(a_1\ a_2 \ldots a_i)(S\ a_{i+1}^{m-1}\ a_1^i)^{m-2}\ S(a_{i+1} \ldots a_{m-1}) =$

$x\ S^{m-1}\ x^{-1} = x\ H\ x^{-1}$. If $i = m-1$, then there exist $a_m, \ldots, a_{2m-2} \in A$ such that $([a_1^m], a_{m+1}, \ldots, a_{2m-2})$ is an m-adic identity and hence also

$(a_{2m-2}, [a_1^m], a_{m+1}, \ldots, a_{2m-2})$. Note also that then $([a_{2m-2}a_1^{m-1}], a_m, \ldots, a_{2m-3})$ and $(a_m, \ldots, a_{2m-3}, [a_{2m-2}\, a_1^{m-1}])$ are also m-adic identities. Then $x = a_1\, a_2 \ldots a_{m-1}$, $x^{-1} = a_m\, a_{m+1} \ldots a_{2m-2}$ and $H = S^{m-1} = (a_1^{m-1}(S\, a_m^{2m-2}))^{m-1} = (a_1\, a_2 \ldots a_{m-1}) \cdot (S\, a_m^{2m-3}(a_{2m-2}\, a_1^{m-1}))^{m-2}\, S\, (a_m \ldots a_{2m-2}) = x\, S^{m-1}\, x^{-1} = x\, H\, x^{-1}$. Thus in both cases H is invariant in G.

Proposition 1.32: A sub-m-group S of an m-group A is semi-invariant iff $[a\, S^{m-1}] = [S^{m-1}\, a]$ for all $a \in A$.

Proof. Suppose $[a\, S^{m-1}] = [S^{m-1}\, a]$ for all $a \in A$. Let $H = S^{m-1}$ be the associated subgroup of S and G the containing group of A. Thus in G we have $a\, H\, a^{-1} = H$. Since $a \in A$, then $a^{-1} = (a_2, a_3, \ldots, a_{m-1})$ such that $(a, a_2, \ldots, a_{m-1})$ is an m-adic identity. Since G is generated by A, then H is normal in G and the result follows.

Theorem 1.33: If S is a semi-invariant sub-m-group of A, A/S exists and is an m-group.

Proof. Let H be the associated subgroup of S and G the containing group of A. First note that $A = \cup\{y \mid y \in xH\}$. Consider the set $\{xH \mid x \in A\}$. Since H is normal in G, if $x_1 \neq x_2$, either $x_1H = x_2H$ or they are disjoint. An m-ary operation may now be defined on the set $\{xH \mid x \in A\}$ by defining $(x_1H\, x_2H \ldots x_mH) = [x_1^m]H$. This operation is associative for $((x_1H \ldots x_mH)x_{m+1}H \ldots x_{2m-1}H) =$

$([x_1^m]H\ x_{m+1}H \ldots x_{2m-1}H) = [[x_1^m]\ x_{m+1}^{2m-1}]H = [x_1^i\ [x_{i+1}^{i+m}]\ [x_1^i[x_{i+1}^{i+m}]x_{i+m+1}^{2m-1}]H = (x_1H \ldots x_iH[x_{i+1}^{i+m}]H\ x_{i+m+1}H \ldots x_{2m-1}H) = (x_1H \ldots x_iH(x_{i+1}H \ldots x_{i+m}H)x_{i+m+1}H \ldots x_{2m-1}H)$. Next, it is necessary to show that for any $x\in A$ there exists $a\in A$ such that for $x_1, x_2, \ldots, x_m\in A$, $(x_1H \ldots x_iH\ a\ H\ x_{i+1}H \ldots x_mH) = xH$ or $[x_1^i\ a\ x_{i+1}^m]H = xH$. Since A is an m-group, this can clearly be done. The set $\{xH|x\in A\}$ is A/S.

As opposed to a 2-group, a congruence relation on an m-group need not determine a sub-m-group. Consider the congruence relation of equality. This will determine a sub-m-group if and only if the m-group has an idempotent. The problem will now be formalized and a set of necessary and sufficient conditions given for a congruence relation to determine a sub-m-group.

Definition 1.34: A relation R on A is a subset of A x A.

Definition 1.35: The domain $\pi_1(R)$ of a relation R on A is the set $\pi_1(R) = \{x: (x,y) \in R \text{ for some } y \in A\}$ and its range is the set $\pi_2(R) = \{y: (x,y) \in R \text{ for some } x \in A\}$.

Notation 1.36: If U is a subset of A, denote

$$RU = \pi_1(R \cap (A \times U))$$

$$UR = \pi_2(R \cap (U \times A)).$$

Definition 1.37: A congruence relation R on an m-group A is a relation which is reflexive, symmetric, and transitive, and such that if $(x_1,y_1) \in R, \ldots, (x_m,y_m) \in R$ then $([x_1^m],[y_1^m]) \in R$.

Theorem 1.38: Let R be a congruence relation on an m-group A. Then $R^* = (A \times A) \cup \{(x,y) | x,y \in A^i$ for some $i = 2, \ldots, m$ such that $x = e_1 \ldots e_{i-1}x'$, $y = e_1 \ldots e_{i-1}y'$ implies $(x',y') \in R\}$ is a congruence in the covering group of A.

Proof. Let $(e_1, e_2, \ldots, e_{m-1})$ be an m-adic identity of A. Note that each $x \in G$ is either an element of A or can be uniquely written in the form

$$x = e_1 e_2 \ldots e_i x'$$

for some $x' \in A$ and $i = 1, 2, \ldots, m-2$. For each $x, y \in G$ define $(x,y) \in R^*$ if and only if either $x, y \in A$ or $x, y \in A^i$ for some $i = 2, \ldots, m-1$ such that $x = e_1 e_2 \ldots e_{i-1}x'$ and $y = e_1 e_2 \ldots e_{i-1}y'$ with $(x', y') \in R$. Reflexivity, symmetry, and transitivity of the relation R^* are clear. Suppose $(x_t, y_t) \in R^*$ $(t = 1, 2)$ so that either $x_1, y_1 \in A$ or $x_1 = e_1 e_2 \ldots e_{i-1}x_1'$, $y_1 = e_1 e_2 \ldots e_{i-1}y_1'$ with $(x_1', y_1') \in R$ and either $x_2, y_2 \in A$ or $x_2 = e_1 e_2 \ldots e_{j-1}x_2'$, $y_2 = e_1 e_2 \ldots e_{j-1}y_2'$ with $(x_2', y_2') \in R$. Obviously, $x_1x_2, y_1y_2 \in A^k$ for some $k = 1, 2, \ldots, m-1$. Thus it suffices to consider the case when k is greater than 1. Let $x_1x_2 = e_1 e_2 \ldots e_{k-1}x'$ and $y_1y_2 = e_1 e_2 \ldots e_{k-1}y'$. We shall show $(x', y') \in R$. Observe

$$x' = [e_k^{m-1} e_1^{k-1} x'] = [e_k^{m-1} x_1 x_2] = [e_k^{m-1} e_1^{i-1} x_1' e_1^{j-1} x_2']$$

and $$y' = [e_k^{m-1} e_1^{k-1} y'] = [e_k^{m-1} y_1 y_2] = [e_k^{m-1} e_1^{i-1} y_1' e_1^{j-1} y_2'].$$

Since $(x_1', y_1') \in R$, $(x_2', y_2') \in R$ and R contains the diagonal of R, then $([e_k^{m-1} e_1^{i-1} x_1' e_1^{j-1} x_2'], [e_k^{m-1} e_1^{i-1} y_1' e_1^{j-1} y_2']) \in R$. Whence $(x', y') \in R$; and $(x_1x_2, y_1y_2) \in R^*$. Lastly, we also show that $(x, y) \in R^*$ implies $(x^{-1}, y^{-1}) \in R^*$.

Suppose $(x, y) \in R^*$ so that either $x, y \in A$ or $x = e_1 \ldots e_{i-1}x'$ and $y = e_1 \ldots e_{i-1}y'$ with $(x', y') \in R$. We shall consider two cases. In case $i = m-2$, so that $x = e_1e_2 \ldots e_{m-3}x'$ and $y = e_1e_2 \ldots e_{m-3}y'$ with $(x', y') \in R$, then $x^{-1} = x'_o \in A$ and $y^{-1} = y'_o \in A$. Thus

$$x'_o = [e_1^{m-1}x'_o] = [[e_1^{m-1}x'_o]e_1^{m-3}y'y'_o]$$

$$y'_o = [e_1^{m-1}y'_o] = [e_1^{m-1}[x'_oe_1^{m-3}x'y'_o]].$$

Since $(x', y') \in R$ and R contains the diagonal of A, then $([e_1^{m-1}x'_oe_1^{m-3}x'y'_o], [e_1^{m-1}x'_oe_1^{m-3}y'y'_o]) \in R$. Whence $(x^{-1}, y^{-1}) = (x'_o, y'_o) \in R$. Otherwise, in case $i \neq m-2$, let $x^{-1} = e_1e_2 \ldots e_{m-i-2}x'_o$ and $y^{-1} = e_1e_2 \ldots e_{m-i-2}y'_o$. As in the previous case, from the fact that

$$([e_{m-i-1}^{m-1}e_1^{m-i-2}y'_oe_1^{i-1}x'e_1^{m-i-2}x'_o], [e_{m-i-1}^{m-1}e_1^{m-i-2}y'_oe_1^{i-1}y'e_1^{m-i-2}x'_o]) \in R \text{ and}$$

$$y'_o = [e_{m-i-1}^{m-1}e_1^{m-i-2}[y'_oe_1^{i-1}x'e_1^{m-i-2}x'_o]] \text{ and}$$

$$x'_o = [e_{m-i-1}^{m-1}e_1^{m-i-2}x'_o] = [e_{m-i-1}^{m-1}[e_1^{m-i-2}y'_oe_1^{i-1}y'e_1]e_2^{m-i-2}x'_o]$$

$$= [e_{m-i-1}^{m-1}e_1^{m-i-2}[y'_oe_1^{i-1}y'e_1^{m-i-2}x'_o]],$$ it follows that $(x'_o, y'_o) \in R$ and hence $(x^{-1}, y^{-1}) \in R^*$.

Since $e_1 \ldots e_{m-1}$ is the identity in G and R^* is a congruence in the group G, then $e_1e_2 \ldots e_{m-1}R^*$ is a normal subgroup of G. Observe also that $e_1e_2 \ldots e_{m-1}R^* = (e_1e_2 \ldots e_{m-2}) \cdot (e_{m-1}R)$. If $x \in (e_1e_2 \ldots e_{m-2}) \cdot (e_{m-1}R)$ so that $x = e_1e_2 \ldots e_{m-2}x'$ where $(x', e_{m-1}) \in R$, then $(x, e_1e_2 \ldots e_{m-1}) \in R^*$. Thus, the right side of the above equality is contained in the left. If $(x, e_1e_2 \ldots e_{m-1}) \in R^*$,

then $x = e_1e_2\ldots e_{m-2}x'$ for some $x' \in A$ with $(x', e_{m-1}) \in R$ and hence also $x \in (e_1e_2\ldots e_{m-2})\cdot(e_{m-1}R)$. The above equality is thus demonstrated.

Theorem 1.39: Let R be a congruence relation on an m-group A. The congruence class $e_{m-1}R$ of R is a sub-m-group of A if and only if there are elements $e_1, e_2, \ldots, e_{m-2} \in e_{m-1}R$ such that $(e_1, e_2, \ldots, e_{m-1})$ is an m-adic identity.

Proof. If $e_{m-1}R$ is a sub-m-group of A, then obviously it contains elements $e_1, e_2, \ldots, e_{m-2}$ such that $(e_1, e_2, \ldots, e_{m-1})$ is an m-adic identity.

Conversely, suppose $e_{m-1}R$ contains elements $e_1, e_2, \ldots, e_{m-2}$ such that $(e_1, e_2, \ldots, e_{m-1})$ is an m-adic identity. By the preceding theorem, R and $(e_1, e_2, \ldots, e_{m-1})$ determine a congruence relation R* on the covering group G of A whose congruence class of the identity $e_1e_2\ldots e_{m-1}R^* = (e_1e_2\ldots e_{m-2})\cdot(e_{m-1}R) = N$ is a normal subgroup of G. We shall show that $e_{m-1}R = (e_{m-1}e_1\ldots e_{m-2})\cdot(e_{m-1}R) = e_{m-1}N$ is a sub-m-group of A. It is clearly a subset of A. The closure of $e_{m-1}R = e_{m-1}N$ is also clear, for, if $x_i \in e_{m-1}R$ for $i = 1, 2, \ldots, m$, so that $(x_i, e_{m-1}) \in R$, then $([x_1^m], [(e_{m-1})^m]) \in R$. Since also $(e_i, e_{m-1}) \in R$ $(i = 1, 2, \ldots, m-1)$, then $(e_{m-1}, [(e_{m-1})^m]) = ([e_1^{m-1}e_{m-1}], [(e_{m-1})^m]) \in R$. Hence $([x_1^m], e_{m-1}) \in R$ and therefore $[x_1^m] \in e_{m-1}R$. Finally, let $x_1, x_2, \ldots, x_{m-1} \in e_{m-1}R = e_{m-1}N$. Then $x_i = e_{m-1}n_i$ $(i = 1, \ldots, m-1)$ and

$[(e_{m-1}n_1)(e_{m-1}n_2)\ldots(e_{m-1}n_{m-1})(e_{m-1}N)] = e_{m-1}n_1e_{m-1}n_2\ldots e_{m-1}n_{m-1}e_{m-1}N = [(e_{m-1})^m]n_1'n_2'\ldots n_{m-1}'N$ for some n_1', n_2', ..., $n_{m-1}' \in N$. From the steps above recall $(e_{m-1}, [(e_{m-1})^m]) \in R$ or $[(e_{m-1})^m] \in e_{m-1}R = e_{m-1}N$ so that $[(e_{m-1})^m] = e_{m-1}n$ for some $n \in N$. Hence $[x_1x_2\ldots x_{m-1}(e_{m-1}N)] = [(e_{m-1}n_1)(e_{m-1}n_2)\ldots(e_{m-1}n_{m-1})(e_{m-1}N)] = [(e_{m-1})^m]n_1'n_2'\ldots n'_{m-1}N = e_{m-1}(nn_1'n_2'\ldots n_{m-1}'N) = e_{m-1}N$. In an analogous manner $[(e_{m-1}N)x_1x_2\ldots x_{m-1}] = e_{m-1}N$. This completes the proof that $e_{m-1}N = e_{m-1}R$ is a sub-m-group of A.

Corollary 1.40: If R is a congruence relation on an m-group A with idempotent e, then eR is a sub-m-group of A.

Proof. The proof follows immediately from 1.38.

Under certain circumstances it may be possible to reduce, say, a 5-group to a 3-group immediately by re-defining the operation. E. L. Post [15] summarized this possibility in the following. Similar work has also been done by Hosszu [9].

Definition 1.41: An n-1 tuple $(a_1, a_2, \ldots, a_{n-1})$ will be said to be commutative with an element x if and only if

$$(x, a_1, a_2, \ldots, a_{n-1}) \equiv (a_1, a_2, \ldots, a_{n-1}, x).$$

Proposition 1.42: An m-group is reducible to an n-group, $m = k(n-1) + 1$, k a positive integer, if and only if there exists an n-1 tuple $(a_1, a_2, \ldots, a_{n-1})$ which commutes with every element of the m-group and such that

$(a_1, a_2, \ldots, a_{n-1}, a_1, a_2, \ldots, a_{n-1}, \ldots, a_1, a_2, \ldots, a_{n-1})$,

with the n-1 tuple $(a_1, a_2, \ldots, a_{n-1})$ repeated k times, is an m-adic identity.

The n-adic operation may be defined as $(x_1 \ldots x_n) = [x_1 \ldots x_n \, a_1^{n-1} \ldots a_1^{n-1}]$ where the right-hand side of the above equation is the ordinary m-group operation, and a_1^{n-1} occurs k-1 times. $(k-1)(n-1)+n = kn-n+1-k+n = kn-k+1 = k(n-1)+1$ so the operation is properly defined.

Proof. If there exists an n-1 tuple $(a_1, a_2, \ldots, a_{n-1})$ satisfying the stated properties, Condition 2 of Theorem 1.26 is immediately satisfied since the m-group has this property. Furthermore, the operation is associative for $((x_1^n)\, x_{n+1}^{2n-1}) = [(x_1^n)\, x_{n+1}^{2n-1}\, a_1^{n-1} \ldots a_1^{n-1}] = [[x_1^n\, a_1^{n-1} \ldots a_1^{n-1}]\, x_{n+1}^{2n-1}\, a_1^{n-1} \ldots a_1^{n-1}] = [x_1 [x_2^{n+1}\, a_1^{n-1} \ldots a_1^{n-1}]\, x_{n+2}^{2n-1}\, a_1^{n-1} \ldots a_1^{n-1}] = (x_1 (x_2^{n+1})\, x_{n+2}^{2n-1}) = \ldots = [x_1^{n-1}\, [x_n^{2n-1}\, a_1^{n-1} \ldots a_1^{n-1}]\, a_1^{n-1} \ldots a_1^{n-1}] = (x_1^{n-1}(x_n^{2n-1}))$.

Next, if the m-group is reducible to an n-group, the n-group has an n-adic identity. $(a_1, a_2, \ldots, a_{n-1}, a_1, a_2, \ldots, a_{n-1}, a_1, a_2, a_{n-1}, \ldots, a_1, a_2, \ldots, a_{n-1})$ k times will then be an m-adic identity and that $(a_1, a_2, \ldots, a_{n-1})$ commutes with every element of A is clear by Proposition 1.13 and the fact that A reduces to an n-group.

Corollary 1.43: Every commutative m-group with idempotent reduces to a 2-group.

Proof. It is immediately apparent that the conditions of the preceding theorem are satisfied.

Example 1.6 is an example of an m-group, $m > 2$, that

does not reduce to a 2-group unless $h \equiv 0\ (m-1)$ since commutativity implies that there must exist an integer n such that $(n, n, \ldots, n)$ is an m-adic identity, i.e., $(m-1)n+x+h = x$ or $n = -h/(m-1)$ which is integral only if $h \equiv 0\ (m-1)$.

Observe that if $h = -n(m-1)$ so that $[x_1^m] = \Sigma_{i+1}^m x_i - n(m-1)$ and n is idempotent, then $(Z, +) \cong (Z, \oplus)$ where $x \oplus y = x + y + n(m-2) - n(m-1) = x+y-n$.

The required isomorphism is given by the bijection $f:Z \longrightarrow Z$ with $f(x) = x+n$. $f(x+y) = x+y+n = (x+n) + (y+n) - n = f(x) \oplus f(y)$.

We next state and prove several elementary properties of m-groups which will be needed in the sequel.

Theorem 1.44: Let Γ be an indexing set and suppose that for each $\alpha \in \Gamma$ there exists an m-group A_α. Then:

(i) With coordinate-wise multiplication, the product $P_{\alpha\in\Gamma} A_\alpha = A$, is an m-group.

(ii) If Λ is an indexing set and for some $\alpha \in \Gamma$ $\{S_\tau \mid \tau \in \Lambda\}$ is a collection of sub-m-groups of A_α with a non-empty intersection, $S = \bigcap_{\tau\in\Lambda} S_\tau$ is a sub-m-group of A_α.

(iii) If $f:A \longrightarrow A$ is a homomorphism, $f(A)$ is a sub-m-group of A.

Proof. (i) Since we have unique solvability and associativity in each coordinate and the operation is coordinate-wise, the proof is clear.

(ii) Associativity is clear. Hence, let $s_1, \ldots s_m, x \in S$. Since $s_1, \ldots s_m, x \in S_\tau$ for each $\tau \in \Lambda$, and each S_τ is a sub-m-group, there exists $a_\tau \in S_\tau$ such that

$[s_1^{i-1}\ a_\tau\ s_{i+1}^m] = x$, and all these a_τ must be equal by the unique solvability in A_α, $a_\tau = a \in S_\tau$ for all τ. Hence $a \in S$ and S is a sub-m-group of A_α.

(iii) Associativity is clear. Let $y, y_1, \ldots y_m \in f(A_\alpha)$. Hence there exists $x, x_1, \ldots, x_m \in A_\alpha$ with $f(x) = y$ and $f(x_i) = y_i$ $i = 1, \ldots, m$. There exists $a \in A_\alpha$ such that $[x_1^{i-1}\ a\ x_{i+1}^m] = x$ and for this a, $f([x_1^{i-1}\ a\ x_{i+1}^m]) = f(x)$ or $[f(x_1)\ldots f(x_{i-1})f(a)f(x_{i+1})\ldots f(x_m)] = f(x)$. So there exists $b = f(a) \in f(A_\alpha)$ such that $[y_1^{i-1}\ b\ y_{i+1}^m] = y$ and we have unique solvability in $f(A_\alpha)$. Note that if $y_1, \ldots, y_m \in f(A_\alpha)$, $[y_1^m] \in f(A_\alpha)$ for in the notation above, $[y_1^m] = [f(x_1)\ldots f(x_m)] = f([x_1^m]) \in f(A_\alpha)$.

CHAPTER II

TOPOLOGICAL m-GROUPS

In this chapter we derive some of the properties of topological m-groups and quotient m-groups.

The following definition is due to F. M. Sioson.

Definition 2.1: A topological m-group $(A, [\,], \tau)$ is an m-group $(A, [\,])$ together with a topology τ on A under which the functions f and g defined by $f(x_1, x_2, \ldots, x_m) = [x_1^m]$ and $g(x_1, x_2, \ldots, x_{m-2}) = x_{m-1}$, where $(x_1, x_2, \ldots, x_{m-1})$ is an m-adic identity, are continuous.

Proposition 2.2: If $(A, [\,], \tau)$ is a topological m-group, then for any $k = 1, 2, \ldots$, $(A, [\,], \tau)$ is also a topological $k(m-1)+1$ group $(A, (), \tau)$ when we define $(a_1^{k(m-1)+1}) = [[\cdots[a_1^m]\cdots]a_{(k-1)(m-1)+1}^{k(m-1)+1}]$.

Proof. The proof is clear.

Example 2.3: Let R^- be the set of negative real numbers with $[\,]:(R^-)^3 \longrightarrow R^-$ defined by $[xyz] = x \cdot y \cdot z$, the ordinary product of x, y, and z. Let the topology on R^- be the usual topology τ on R restricted to R^-. $(R^-, [\,], \tau)$ is a topological 3-group.

Example 2.4: Let $S' = \{z \mid z = e^{ix},\ 0 \leq x < 2\pi\}$. Define $[\,]:(S')^m \longrightarrow S'$ as follows: If $z_1, \ldots, z_m \in S'$ and

$z_1 = e^{ix_1}$, ..., $z_m = e^{ix_m}$, then $[z_1^m] = e^{i(x_1+\cdots+x_m)}$. If τ is the usual topology on S', then (S', [], τ) is a topological m-group.

Example 2.5: If (G, $\cdot$, τ) is an ordinary topological group, and if h is in the center of G, (G, [], τ) will be a topological m-group if []: $G^m \longrightarrow G$ is defined as $[x_1^m] = x_1 \cdot x_2 \cdots x_m \cdot h$.

Theorem 2.6: The function h: A $\longrightarrow$ A defined by $h(x) = [a_1^{i-1} x a_{i+1}^m]$ is a homeomorphism for each choice of i = 1, ..., m and elements a_1, ..., $a_m \in A$ (where by convention $[a_1^{i-1} x a_{i+1}^m]$ is $[x a_2^m]$ when i = 1 and $[a_1^{m-1} x]$ when i = m).

Proof. The function h is the restriction of the map f to the subset $\{a_1\}\times\ldots\times\{a_{i-1}\}\times A \times \{a_{i+1}\}\times\ldots\times\{a_m\}$ of A x A x...x A (m times) and hence is continuous. Let b_1, b_2, ..., $b_{m-1} \in A$ such that (a_1, a_2, ..., a_{i-1}, b_i, ..., b_{m-1}) and (b_1, b_2, ..., b_{i-1}, a_{i+1}, ..., a_m) are m-adic identities and define $k(x) = [b_1^{i-1} x b_i^{m-1}]$. As before, k is continuous. Note however, that hk = identity = kh. Thus h and k are inverses of each other and are both bijective. Whence, h is a homeomorphism.

Corollary 2.7: Every topological m-group is homogenous.

Proof. If a and b are any two elements of a topological m-group, then for elements a_1, ..., a_m in the m-group such that $[a_1^{i-1} a a_{i+1}^m] = b$, the map $h(x) = [a_1^{i-1} x a_{i+1}^m]$ is a homeomorphism that takes a to b.

Proposition 2.8: Let A be a topological m-group and let $A_1, A_2, \ldots, A_m$ be any m subsets of A. If A_i is open for some i, then $[A_1, A_2, \ldots, A_m]$ is open. If $A_1, \ldots, A_m$ are compact, then $[A_1 \ldots A_m]$ is compact.

Proof. If A_i is open, then by Theorem 2.6 $[a_1 \cdots a_{i-1}\ A_i\ A_{i+1} \cdots a_m]$ is open, and $[A_1 \cdots A_m] = \cup\{[a_1 \cdots a_{i-1}\ A_i\ a_{i+1} \cdots a_m] \mid a_i \in A_m\}$ is open. Since $f:(x_1, \ldots, x_m) \longrightarrow [x_1^m]$ is continuous, if each A_i is compact, $A_1 \times A_2 \times \cdots \times A_m$ is compact, so $[A_1 A_2 \ldots A_m]$ being the continuous image of a compact set is compact.

Proposition 2.9: Let A be an m-group with idempotent e. Then for any neighborhood U of e there exists a neighborhood V of e such that $[V^m] \subset U$.

Proof. Since $f:(e,\ldots,e) \longrightarrow e$ is continuous if U is any neighborhood of e, there exists $U_1 \ldots U_m$ neighborhoods of e with $[U_1 \ldots U_m] \subset U$.

Let $V = \bigcap_{i=1}^{m} U_i$ and we see that $[V^m] \subset U$.

Proposition 2.10: Let A be a topological m-group with S a sub-m-group of A. Then S with the relative topology is a topological m-group.

Proof. The mappings f and g in Definition 2.1 are continuous and hence the restrictions of f and g to S are continuous.

Proposition 2.11: Let A be a topological m-group and let $A_1, A_2, \ldots, A_m$ be subsets of A. Then

(i) $[\bar{A}_1 \ldots \bar{A}_m] \subset [A_1 \ldots A_m]^-$

(ii) $(\bar{A}_1, \ldots, \bar{A}_{m-2})^{-1} \subset ((A_1, \ldots, A_{m-2})^{-1})^-$

(iii) $[x_1^{i-1} \ldots \bar{A}_j\ x_{i+1}^m] = [x_1^{i-1}\ A_j\ x_{i+1}^m]^-$.

Proof. (i) It is known that for any continuous function f, $f(\bar{A}) \subset \overline{f(A)}$.

(ii) Same as (i).

(iii) By Theorem 2.6 $f{:}x \longrightarrow [x_1^{i-1}\ x\ x_{i+1}^m]$ is a homeomorphism.

Proposition 2.12: If H is a sub-m-semigroup, sub-m-group, or semi-invariant sub-m-group of a topological m-group A, then H^- is, respectively, each of these.

Proof. Let H be a sub-m-semigroup of A. Then $[H^m] \subset H$ and by Proposition 2.11 (i) $[(H^-)^m] \subset [H^m]^- \subset H^-$.

Next, let H be a sub-m-group of A. Then, as in the first part of this proof, $[H^m]^- \subset H^-$. Since $(H, H, \ldots H)^{-1} \subset H$, by Proposition 2.11 (ii) $(H^-, H^-, \ldots H^-)^{-1} \subset ((H, H, \ldots H)^{-1})^- \subset H^-$.

H is defined to be an invariant sub-m-group iff for each m-adic identity $(e_1, e_2, \ldots, e_{m-1})$, $[e_1^i\ H\ e_{i+1}^{m-1}] = H$ and by Proposition 2.11(iii) $[e_1^i\ H^-\ e_{i+1}^{m-1}] = [e_1^i\ H\ e_{i+1}^{m-1}]^- = H^-$. The proof is similar for semi-invariant sub-m-groups.

Proposition 2.13: A sub-m-group H of a topological m-group A is open iff its interior is not empty. Every open sub-m-group is closed.

Proof. Suppose the interior is not empty and let e_{m-1} be an interior point of H. Then there exists an open

neighborhood V of e_{m-1} with $V \subset H$. Let $e_1, \ldots, e_{m-2}$ be a collection of m-2 elements from H such that $(e_1, \ldots, e_{m-1})$ is an m-adic identity. (Such a collection exists since H is a sub-m-group.) Hence for any $h \in H$, $[h\ e_1^{m-1}] = h$, so $[H^{m-1}\ V] = H$ and by Proposition 2.8, H is open. By Theorem 2.6, $[x_1^{m-1}\ H]$ is open for any choice of $x_1, \ldots, x_{m-1} \in A$. Hence, $\cup \{[x_1^{m-1}\ H] | h \notin [x_1^{m-1}\ H]$ for any $h \in H\}$ is open and is $A \backslash H$. Hence, H is closed.

Proposition 2.14: A sub-m-group H of a topological m-group A is discrete iff it has an isolated point.

Proof. Suppose H has an isolated point x. Then there exists an open set $U \subset A$ such that $U \cap H = \{x\}$. Let $y \in H$. Since H is a sub-m-group of A, there exists $x_1, x_2, \ldots, x_{m-1} \in H$ such that $[x_1^{m-1}\ x] = y$. Since U is open about x, $[x_1^{m-1}\ U]$ is open about y by Theorem 2.6. Since $U \cap H = \{x\}$, $\{y\} \in [x_1^{m-1}\ U] \cap H$. If $y \neq y_o \in [x_1^{m-1}\ U] \cap H$, $y_o \in [x_1^{m-1}\ U]$ which implies that for some $x_o \in U$, $y_o = [x_1^{m-1}\ x_o]$. Now $x_1, x_2, \ldots, x_{m-1} \in H$, $y_o \in H$ and H being a sub-m-group implies that $x_o \in H$, and therefore, that $x_o \in U \cap H$. But $x_o \neq x$ by uniqueness and the choice of $y_o \neq y_1$, a contradiction. Hence, $[x_1^{m-1}\ U] \cap H = \{y\}$ and H is discrete. The converse is clear, for if it is discrete, all of its points are isolated.

Definition 2.15: A relation R on a topological space A is lower semicontinuous iff UR is open for every open set U in A.

Definition 2.16: A relation R on a topological space A is said to be closed iff it is a closed subset of A x A under its product topology.

Definition 2.17: A relation R on a topological space A is upper (lower) semiclosed iff $xR = \{x\}R$ ($Rx = R\{x\}$) is closed for every choice of $x \in A$.

The following theorem has been proved for general algebraic systems in which the congruences commute by Mal'cev [12, p. 136]. It can be shown that the congruences in an m-group commute but we will prove the theorem directly.

Theorem 2.18: Any congruence R of a topological m-group A is lower semicontinuous.

Proof. Let U be any open subset of A and suppose UR is not open; that is to say, there is an $x \in UR$ such that for any neighborhood V of x, we have $V \cap (A \backslash UR) \neq \square$. Since $x \in UR$, then there is a $y \in U$ such that $(x, y) \in R$. Let $x_2, x_3, \ldots, x_{m-1}$ be elements in A such that $(x, x_2, \ldots, x_{m-1})$ is an m-adic identity so that $[xx_2^{m-1}y] \in U$. By the continuity of the m-ary operation on A, then there exists an open set V containing x such that

$$[Vx_2^{m-1}y] \subset U.$$

By hypothesis, since UR is not open, then there is a $v \in V$ such that $v \notin UR$. Since $(x,y) \in R$, $(v,v) \in R$, and $(x_i, x_i) \in R$ for all $i = 2, \ldots, m-1$, then $([vx_2^{m-1}y], [vx_2^{m-1}x]) \in R$. However, $(x_2, \ldots, x_{m-1}, x)$, being a cyclic permutation of

an m-adic identity, is also an m-adic identity so that $([vx_2^{m-1}y], v) \in R$. Note $[vx_2^{m-1}y] \in U$ and hence $([vx_2^{m-1}y], v) \in R \cap (U \times A)$. This implies that $\pi_2([vx_2^{m-1}y], v) = v \in UR$, which is a contradiction. Therefore, UR must be open.

Definition 2.19: Let $(A, [\,], \tau)$ be a topological m-group and let R be an equivalence relation on A. Define $n: A \longrightarrow A/R$ by $n(a) = aR$ for each $a \in A$. n will be called the natural map. Let $\mathfrak{U}$ be the family of subsets of A/R defined by $U \in \mathfrak{U}$ iff $n^{-1}(U)$ is open in A.

Remark 2.20: If R is a congruence and $U \in \mathfrak{U}$, then U may be expressed as $\{xR | x \in T \in \tau\}$ for if $U \in \mathfrak{U}$, set $T = n^{-1}(U)$. Conversely, if T is open in A, $n(T) \in \mathfrak{U}$ for $n^{-1}(n(T)) = TR$ which is open in A since R is lower semi-continuous.

Theorem 2.21: The family of sets $\mathfrak{U}$ in Definition 2.19 is a topology for A/R. The mapping n is continuous and $\mathfrak{U}$ is the strongest topology on A/R under which n is continuous.

Proof. Let $\{uR | u \in T_\lambda\}_{\lambda \in \Lambda}$ $T_\lambda \in \tau$ be an arbitrary collection of sets in $\mathfrak{U}$. Then $\bigcup_{\lambda \in \Lambda} \{uR | u \in T_\lambda\} = \{uR | u \in \bigcup_{\lambda \in \Lambda} T_\lambda\} \in \mathfrak{U}$ since $\bigcup_{\lambda \in \Lambda} T_\lambda$ is open in A. If $\{uR | u \in T_i\}_{i=1}^{n}$ $T_i \in \tau$ is a finite collection of members of $\mathfrak{U}$, $\bigcap_{i=1}^{n} \{uR | u \in T_i\} = \{uR | u \in \bigcap_{e=1}^{n} T_i\}$ which is in $\mathfrak{U}$ since $\bigcap_{i=1}^{n}$ is open in A. Hence $\mathfrak{U}$ is a topology on A/R. It is

clear that n is continuous. Next, let $\mathfrak{V}$ be another topology on A/R such that $\mathfrak{U} \subset \mathfrak{V}$. Let $F \in \mathfrak{V}$ be an $\mathfrak{V}$-open set which is not $\mathfrak{U}$-open and suppose n is continuous under the topology $\mathfrak{V}$. Then $n^{-1}(F)$ is open in A, so suppose $n^{-1}(F) = T \in \tau$. Then n(T) is an element of $\mathfrak{U}$ by the remark, so n(T) is $\mathfrak{U}$-open and $n(T) = n(n^{-1}(F)) = F$, i.e., F is $\mathfrak{U}$-open, a contradiction. So n is not continuous under the topology $\mathfrak{V}$ and $\mathfrak{U}$ is the strongest topology under which n is continuous.

Proposition 2.22: The natural mapping of A onto A/R is open.

Proof. By Remark 2.20, if T is open in A, n(T) is open in A/R.

Proposition 2.23: Let A be a topological m-group with congruence relation R on it. If aR is compact for some $a \in A$, xR is compact for all $x \in A$.

Proof. Let $x \in A$ and choose $a_1, \ldots, a_{m-1} \in A$ such that $[a_1^{m-1}\ a] = x$. Observing that $[a_1^{m-1}\ (aR)] = \{z|([a_1^{m-1}\ q], z) \in R \text{ and } (q,a) \in R\} = \{z|([a_1^{m-1}\ a], z) \in R\} = [a_1^{m-1}\ a]R = xR$ we see that xR is compact by Theorem 2.6 being the continuous image of a compact set.

Proposition 2.24: Let A be a topological m-group with a congruence relation R on A. Then A/R is discrete if and only if aR is open in A for some $a \in A$.

Proof. Suppose aR is open for some $a \in A$. For any $x \in A$, let $a_1, \ldots, a_{m-1}$ be chosen so that $[a_1^{m-1}\ a] = x$. Then $[a_1^{m-1}\ (aR)] = xR$ which is open by Theorem 2.6. Hence, if aR is open for some $a \in A$, xR is open for any $x \in A$.

By Proposition 2.22, $\{yR|y \in xR\} = xR \in A/R$ is open. Hence, if xR is an open subset of A, $\{xR\}$ is open as an element of A/R, so A/R is discrete. If A/R is discrete, $\{aR\}$ is open in A/R, so $n^{-1}\{aR\} = aR$ is open in A.

Proposition 2.25: If an equivalence relation R on a T_o topological space A is closed, then A/R is a T_1 topological space under its quotient topology.

Proof. From Wallace's Algebraic Topology Notes [26, Cor. 3, p. 8] we know that if Y is compact, then the projection map π: A x Y $\longrightarrow$ Y is closed. If R is closed, then $R \cap (A \times \{y\})$ is also closed for any $y \in A$. Hence, $\pi_1(R \cap (A \times \{y\}))$ is closed and A/R is T_1.

Theorem 2.26: If h: $\tilde{A} \longrightarrow$ A is a topological (continuous) homomorphism between two topological m-groups and A is T_o, then the congruence relation $R = h \cdot h^{-1}$ is lower semicontinuous, lower and upper semiclosed, and closed.

Proof. Lower semicontinuity follows by a previous theorem. If Δ denotes the diagonal relation on A, then note that $h \cdot h^{-1} = (h \times h)^{-1}(\Delta)$. A being T_o implies it is in fact T_2 and hence Δ is closed in A x A. Hence, $h \cdot h^{-1}$ is also closed. By the same reasoning as in the proof of Proposition 2.25, $Ry = \pi_1((h \times h)^{-1}(\Delta) \cap (A \times \{y\}))$ and $yR = \pi_2((h \times h)^{-1}(\Delta) \cap (\{y\} \times A))$ are closed for each $y \in A$ and hence R is both lower and upper semiclosed.

Theorem 2.27: If h: $\tilde{A} \longrightarrow$ A is an open continuous epimorphism of T_o topological m-groups, then A/R where $R = h \cdot h^{-1}$ is iseomorphic to A under the natural mapping.

Proof. The algebraic isomorphism between $\tilde{A}/R$ and A follows from general algebra. $\tilde{A}/R$ and A possess precisely the same open sets which are images of open sets in $\tilde{A}$. Hence, they are homeomorphic.

Theorem 2.28: Let A be a compact (locally compact) topological m-group. If R is a congruence relation on A, then A/R is compact (locally compact).

Proof. Since n is continuous, A/R is the continuous image of a compact set and hence compact. Next, let $\{xR\} \in A/R$ and let U be an open neighborhood of x in A such that $\bar{U}$ is compact. Then n(U) is open about {xR} in A/R and $n(U)^{-}$ is compact since n is continuous [8, 3.13].

Definition 2.29: A topological space X has the fixed point property if and only if for each continuous map $h: X \longrightarrow X$ there exists $x \in X$ such that $h(x) = x$.

Theorem 2.30: Let A be a topological m-group. Then A does not have the fixed point property.

Proof. Define a function $h: A \longrightarrow A$ by choosing $x_1, \ldots, x_{m-1} \in A$ such that $(x_1, \ldots, x_{m-1})$ is not an m-adic identity and letting $h(a) = [x_1^{m-1}\ a]$. By Corollary 2.7, h is continuous and suppose h(a) = a for some $a \in A$. Then $[x_1^{m-1}\ a] = a$, so by Propositions 1.12 and 1.13, $(x_1, \ldots, x_{m-1})$ is an m-adic identity contradicting the choice of $(x_1, \ldots, x_{m-1})$.

Remark 2.31: If A is a topological m-group, then A is not homeomorphic to $[0,1] \times \ldots \times [0,1] = [0,1]^n$ for any n, nor is A homeomorphic to the Tychonoff cube since both

have the fixed point property [5, p. 301].

Theorem 2.32: Let A be a topological m-group, and for some $x \in A$, let C be the component of x. Then, if $[x_1^{m-1} x] = y$, $[x_1^{m-1} C]$ is the component of y.

Proof. Let x and y be given as in the statement of the theorem and let $x_1, \ldots, x_{m-1} \in A$ be chosen so that $[x_1^{m-1} x] = y$. Then $y \in [x_1^{m-1} C]$ and $[x_1^{m-1} C]$ is connected and closed by Corollary 2.7. $[x_1^{m-1} C]$ is the component of y, for if not, let K be the component of y. Then $[x_1^{m-1} C] \subsetneq K$. Let $x_{i,1}, \ldots x_{i,m-2} \in A$ be chosen such that $(x_{i,1}, \ldots, x_{i,m-2}, x_i)$ is an m-adic identity, $i = 1, \ldots, m-1$. Then $[x_{m-1,1} \cdots x_{m-1,m-2} \cdots x_{1,1} \cdots x_{1,m-2}\ x_1 x_2 \cdots x_{m-1}\ C] = C \subsetneq [x_{m-1,1} \cdots x_{m-1,m-2} \cdots x_{1,1} \cdots x_{1,m-2}\ K]$ which is closed and connected, contradicting the assumption that C is the component of x.

CHAPTER III

AN EMBEDDING THEOREM

In this chapter we will prove the topological version of the Post Coset Theorem by showing that each topological m-group can be considered as the coset of a normal (in the group sense), open, and closed subgroup of an ordinary topological group. Several consequences of this theorem will also be exhibited.

Proposition 3.1: Let $(A, [\,])$ be an m-group with A^{m-1} as its associated group according to the Post Coset Theorem. If $(e_1, e_2, \ldots, e_{m-1})$ is any m-adic identity of A so that $e_1, e_2, \ldots, e_{m-1} \in A$, then for each $x \in A^{m-1}$ and any fixed index $i = 1, \ldots, m-1$, there exists an element $a \in A$ such that

$$e_1 \ldots e_{i-1} a e_{i+1} \ldots e_{m-1} = x.$$

Proof. Since $x \in A^{m-1}$, then $x = a_1 a_2 \ldots a_{m-1}$ for some $a_1, a_2, \ldots, a_{m-1} \in A$. For any $a_m \in A$, there exists, by definition of an m-group, a unique $a \in A$ such that $[e_1^{i-1} a e_{i+1}^{m-1} a_m] = [a_1^m]$. Considered in the containing group of A, the equality above reduces to $e_1 \ldots e_{i-1} a e_{i+1} \ldots e_{m-1} a_m = a_1 a_2 \ldots a_{m-1} a_m = x a_m$, and hence, $e_1 \ldots e_{i-1} a e_{i+1} \ldots e_{m-1} = x$.

Proposition 3.2: Let $(e_1, e_2, \ldots, e_{m-1})$ be an m-adic identity of a topological m-group $(A, [\,], \tau)$ with $\mathfrak{U}(e_{m-1})$ a local open basis at e_{m-1}. Then $\mathfrak{U}(e_1e_2\ldots e_{m-1}) = \{e_1e_2\ldots e_{m-2}U \mid U \in \mathfrak{U}(e_{m-1})\}$ is a local open basis of the identity $e = e_1e_2\ldots e_{m-1}$ of the containing group of A under some topology.

Proof. It is sufficient to show closure under intersection. If $U_1, U_2 \in \mathfrak{U}(e_{m-1})$, then clearly we have $e_1e_2\ldots e_{m-2}U_1 \cap e_1e_2\ldots e_{m-2}U_2 = e_1e_2\ldots e_{m-2}(U_1 \cap U_2)$.

Remarks. Preparatory to the demonstration that $\mathfrak{U}(e_1e_2\ldots e_{m-1})$ defines a topology that converts the containing group of A into a topological group, we shall prove a series of Lemmata.

Lemma 3.3: Let $(e_1, e_2, \ldots, e_{m-1})$ be an m-adic identity of a topological m-group $(A, [\,], \tau)$. For any $i = 1, \ldots, m-2$ and each open set V containing e_i, there exists a basis element $U \in \mathfrak{U}(e_{m-1})$ such that $e_1e_2\ldots e_{m-2}U \subset e_1\ldots e_{m-1}Ve_{i+1}\ldots e_{m-1}$.

Proof. For each $v \in V$ and fixed element $x_1 \in A$, there exists, by definition of an m-group, an element $w \in A$ such that

$$[e_1^{m-2}wx_1] = [e_1^{i-1}ve_{i+1}^{m-1}x_1].$$

Define two functions h and k by $h(v) = [e_1^{i-1}ve_{i+1}^{m-1}x_1]$ and $k(w) = [e_1^{m-2}wx_1]$. Let $(x_1, x_2, \ldots, x_{m-1})$ be an m-adic identity.

Then

$$W = \{w \mid [e_1^{m-2} w x_1] = [e_1^{i-1} v e_{i+1}^{m-1} x_1] \text{ for some } v \in V\} =$$

$$\{w \mid [e_{m-1}[e_1^{i-1} v e_{i+1}^{m-1} x_1] x_2^{m-1}] = w \text{ for some } v \in V\} =$$

$$[e_{m-1}\, e_1^{i-1} V e_{i+1}^{m-1} x_1\, x_2^{m-1}] = [e_{m-1} h(V) x_2^{m-1}] = k^{-1}h(V)$$

is an open subset of A containing e_{m-1} since both h and k are homeomorphisms by Theorem 2.6.

Since $\mathfrak{U}(e_{m-1})$ is a local basis at e_{m-1}, then there exists a $U \in \mathfrak{U}(e_{m-1})$ such that $U \subset W$. Whence $[e_1^{i-1} V e_{i+1}^{m-1} x_1] = [e_1^{m-2} W x_1] \supset [e_1^{m-2} U x_1]$. Considered in the containing group this gives $e_1 \cdots e_{i-1} V e_{i+1} \cdots e_{m-1} x_1 \supset e_1 \cdots e_{m-2} U x_1$ and hence $e_1 \cdots e_{i-1} V e_{i+1} \cdots e_{m-1} \supset e_1 \cdots e_{m-2} U$.

Notation. In Lemmas 3.4 and 3.5 let f and g be the functions of Definition 2.1.

Lemma 3.4: For each $U \in \mathfrak{U}(e_{m-1})$, there exists a $V \in \mathfrak{U}(e_{m-1})$ such that $(e_1 e_2 \cdots e_{m-2} V)(e_1 e_2 \cdots e_{m-2} V) \subset e_1 e_2 \cdots e_{m-2} U$.

Proof. Let $U \in \mathfrak{U}(e_{m-1})$ and define a function $h: A \times A \longrightarrow A$ by $h(x,y) = [x e_1^{m-2} y]$. Being a restriction of the continuous function f on $A \times \{e_1\} \times \cdots \times \{e_{m-2}\} \times A$, h is therefore also continuous. $h^{-1}(U)$ is thus an open subset of $A \times A$. Since the first and second projection maps π_1 and π_2 are open maps, then $V = \pi_1 h^{-1}(U) \cap \pi_2 h^{-1}(U)$ is also open. It is also non-empty, since $h(e_{m-1}, e_{m-1}) = [e_{m-1} e_1^{m-1}] = e_{m-1} \in U$ and therefore $e_{m-1} \in V$. We now claim $(e_1 e_2 \cdots e_{m-2} V)^2 \subset e_1 e_2 \cdots e_{m-2} U$; for, if $v_1, v_2 \in V$, then

$h(v_1,v_2) \in U$ and hence $(e_1e_2\ldots e_{m-2}v_1)(e_1e_2\ldots e_{m-2}v_2) = e_1e_2\ldots e_{m-2}[v_1e_1^{m-2}v_2] = e_1e_2\ldots e_{m-2}h(v_1, v_2) \in e_1e_2\ldots e_{m-2}U$.

Lemma 3.5: For each $U \in \mathfrak{U}(e_{m-1})$, there exists a $V \in \mathfrak{U}(e_{m-1})$ such that $(e_1e_2\ldots e_{m-2}V)^{-1} \subset e_1e_2\ldots e_{m-2}U$.

Proof. By definition of a topological m-group, $g^{-1}(U)$ is open in A x A x...x A (m-2 times). Since $U \in \mathfrak{U}(e_{m-1})$ and therefore $e_{m-1} \in U$, then $(e_1, e_2, \ldots, e_{m-2}) \in g^{-1}(U)$. Thus, there exist $W_1 \in \mathfrak{U}(e_1)$, $W_2 \in \mathfrak{U}(e_2)$, ..., $W_{m-2} \in \mathfrak{U}(e_{m-2})$ such that $W_1 \times W_2 \times\ldots\times W_{m-2} \subset g^{-1}(U)$. Let $W = W_1W_2\ldots W_{m-2}$. Observe that $W^{-1} \subset U$ in the containing group. Thus

$$(W_1e_2\ldots e_{m-1})^{-1} \subset (We_{m-1})^{-1} = e_{m-1}^{-1}W^{-1} = e_1e_2\ldots e_{m-2}U.$$

By Lemma 3.3, there exists a $V \in \mathfrak{U}(e_{m-1})$ such that

$$e_1e_2\ldots e_{m-2}V \subset W_1e_2\ldots e_{m-1}$$

and therefore

$$(e_1e_2\ldots e_{m-2}V)^{-1} \subset e_1e_2\ldots e_{m-2}U.$$

Lemma 3.6: For each $U \in \mathfrak{U}(e_{m-1})$ and $x \in e_1e_2\ldots e_{m-2}U$, there exists a $V \in \mathfrak{U}(e_{m-1})$ such that $x(e_1e_2\ldots e_{m-2}V) \subset e_1e_2\ldots e_{m-2}U$.

Proof. Let $x = e_1e_2\ldots e_{m-2}u$ for $u \in U$. Define the function h such that $h(v) = [ue_1^{m-2}v]$. Since this is a homeomorphism by Theorem 2.6, then $h^{-1}(U) = W$ is an open and non-empty subset of A. Inasmuch as $e_{m-1} \in W$, then there is an element $V \in \mathfrak{U}(e_{m-1})$ contained in W. Hence $x(e_1e_2\ldots e_{m-2}V) \subset x(e_1e_2\ldots e_{m-2}W) = (e_1e_2\ldots e_{m-2}u)\cdot$

$(e_1e_2\cdots e_{m-2}W) = e_1e_2\cdots e_{m-2}\, ue_1^{m-2}W = e_1e_2\cdots e_{m-2}h(W) = e_1e_2\cdots e_{m-2}U$.

Lemma 3.7: For each $U \in \mathfrak{U}(e_{m-1})$ and any element x in the containing group of A, there exists a $V \in \mathfrak{U}(e_{m-1})$ such that $x(e_1e_2\cdots e_{m-2}V)x^{-1} \subset e_1e_2\cdots e_{m-1}U$.

Proof. Recall that the containing group of A is given by $A \cup A^2 \cup \cdots \cup A^{m-1}$. We consider two cases:

Case I. Suppose $x \notin A^{m-1}$ so that $x = a_1a_2\cdots a_i$. Let $a_{i+1}, \ldots, a_{m-1} \in A$ such that $a_1a_2\cdots a_ia_{i+1}\cdots a_{m-1}$ is the identity element in the containing group. Denote $x^{-1} = a_{i+1}\cdots a_{m-1}$. For any fixed element $a \in A$, let

$$W = \{w \mid [a_1^i[e_1^{m-2}wa_{i+1}]a_{i+2}^{m-1}a] \in [e_1^{m-2}Ua]\}.$$

Since $[e_1^{m-2}Ua]$ is an open subset in A (in fact a basis element at the point $a \in A$), then, being the inverse under a homeomorphism, W is also open. Since $e_{m-1} \in W$, there exists a $V \in \mathfrak{U}(e_{m-1})$ such that W contains V and hence

$$[a_1^i[e_1^{m-2}Va_{i+1}]a_{i+2}^{m-1}a] \subset [a_1^i[e_1^{m-2}Wa_{i+1}]a_{i+1}^{m-1}a] = [e_1^{m-2}Ua].$$

Thus, after simplication in the containing group of A, we obtain $x(e_1e_2\cdots e_{m-2}V)x^{-1} = (a_1\cdots a_i)(e_1e_2\cdots e_{m-2}V)\cdot (a_{i+1}\cdots a_{m-1}) \subset e_1e_2\cdots e_{m-2}U$.

Case II. Suppose $x \in A^{m-1}$ so that $x = a_1a_2\cdots a_{m-1}$ for $a_1, \ldots, a_{m-1} \in A$. Let $a_m, \ldots, a_{2m-2} \in A$ such that $([a_1^m], a_{m+1}, \ldots, a_{2m-2})$ is an m-adic identity and denote

$x^{-1} = a_m a_{m+1} \ldots a_{2m-2}$. For any fixed element $a \in A$, let $W = \{w | [[a_1^{m-1}[e_1^{m-2} w a_m]]\ a_{m+1}^{2m-2} a] \in [e_1^{m-2} U a]\}$. The rest of the proof proceeds as in Case I.

Theorem 3.8: Any topological m-group is the coset of a topological group G by a normal subgroup N with the property that G/N is a finite cyclic and discrete topological group. If $m > 2$ and A does not reduce to a 2-group, then G is disconnected.

Proof. Lemmas 3.4 to 3.7 showed that $\mathfrak{U}(e_1 e_2 \ldots e_{m-1})$ is an open basis at the identity of the containing group G of A which converts G into a topological group [8, 4.5] with basis

$$\{xU: x \in G, U \in \mathfrak{U}(e_1 e_2 \ldots e_{m-1})\} \text{ or}$$

$$\{Ux: x \in G, U \in \mathfrak{U}(e_1 e_2 \ldots e_{m-1})\}.$$

Since A in the containing group is clopen, then each coset of G by N is also clopen (being homeomorphs of A). If $m > 2$, then G is disconnected and G/N is discrete under its quotient topology.

It thus remains to show that the topology defined in G by $\mathfrak{U}(e_1 e_2 \ldots e_{m-1})$ gives rise to the same topology on A when restricted or relativized. By homogeneity of A, it suffices to consider the basis at any point, say $e_{m-1} \in A$. If U is any basis element of A containing e_{m-1}, then $e_1 e_1 e_2 \ldots e_{m-2} U$ is a basis element in G of $e = e_1 e_2 \ldots e_{m-2} e_{m-1}$. Hence, $e_{m-1}(e_1 e_2 \ldots e_{m-2} U) = U$ is also a basis element of G at e_{m-1}. Thus, every basis element of A at e_{m-1} (and hence

at any point) results from a basis element of G. This means that the topology induced on A by the topology in G coincides with the original topology on A.

Corollary 3.9: A T_o topological m-group is always completely regular and hence Hausdorff.

Proof. By Theorem 3.8, any topological m-group A is the coset of a topological group modulo a normal subgroup N which is homeomorphic to A. Since A is T_o, then N is T_o and hence completely regular. Thus A is also completely regular.

Theorem 3.10: Any compact T_o topological m-group A is homeomorphically representable as an m-group of m-adic homeomorphisms.

Proof. Observe that the topological m-group is in fact completely regular and hence Hausdorff (Corollary 3.9).

By the representation theorem for algebraic m-groups, the given m-group A is isomorphic to a sub-m-group of the m-group of m-adic functions $F(S_1, S_2, \ldots, S_{m-1})$ on the sets $S_i = A \times \ldots \times A/\equiv$ (i times). Since A is compact Hausdorff, then $A \times \ldots \times A$ (i times) is compact Hausdorff and hence also S_i under its natural quotient topology. Thus $\bigcup_{i=1}^{m-1} S_i$ is also compact Hausdorff under its sum topology.

Any bijective continuous function on $\bigcup_{i=1}^{m-1} S_i$ is also a homeomorphism and thus the collection of all such functions $H(\bigcup_{i=1}^{m-1} S_i)$ is a group (of homeomorphisms). By a result

of Arens [1, p. 597], since $\bigcup_{i=1}^{m-1} S_i$ is compact, then $H(\bigcup_{i=1}^{m-1} S_i)$ is a topological group under the compact open topology and the operation of composition. Since $F(S_1,S_2,\ldots,S_{m-1}) \subset H(\bigcup_{i=1}^{m-1} S_i)$, then for any $f_1, f_2, \ldots, f_m \in F(S_1, S_2, \ldots, S_{m-1})$, the functions

$$(f_1, f_2, \ldots, f_m) \longrightarrow f_1.f_2\ldots f_m = [f_1^m]$$

and

$$(f_1, f_2, \ldots, f_{m-2}) \longrightarrow (f_1.f_2\ldots f_{m-2})^{-1} = f_{m-2}^{-1}.f_{m-3}^{-1}\ldots f_1^{-1}$$

are continuous. Thus $F(S_1, \ldots, S_{m-1})$ is a topological m-group under the compact open topology (relativized).

Consider now the regular representation

$$h: A \longrightarrow F(S_1,S_2, \ldots, S_{m-1}) \subset H(\bigcup_{i=1}^{m-1} S_i)$$

defined by $h(a) = L_a$. The following Lemma will be needed.

Lemma 3.11: The function $k: \bigcup_{i=1}^{m-1} S_i \longrightarrow \bigcup_{i=1}^{m-1} A^i = G$ such that $k((a_1, a_2, \ldots, a_i)/\equiv) = a_1a_2\ldots a_i$ is a continuous open map onto the containing group G of the topological m-group A.

Proof. Let U be any open subset in G such that $a_1a_2\ldots a_i \in U$. By the continuity of the multiplication in G, then there exist open subsets in G, $a_1 \in V_1$, $a_2 \in V_2$, $\ldots$, $a_i \in V_i$ such that

$$k(V_1 \times V_2 \times\ldots\times V_i/\equiv) = V_1V_2\ldots V_i \subset U.$$

Thus, h is continuous. That h is open is obvious.

Let $L_a \in (C, U)$ or

$\{(a,a_1, \ldots, a_i)/\equiv |$ for some $i = 1, \ldots, m-1$ and

$(a_1, \ldots, a_i)/\equiv \in C\} \subset U$. Applying k on both sides, we have

$\{aa_1a_2\ldots a_i|$ for some $i = 1, \ldots, m-1$ and

$(a_1, \ldots, a_i)/\equiv \in C\} \subset k(U)$.

By Lemma 3.11 $k(C)$ is compact and $k(U)$ is open. Thus, for any fixed index $i = 1, \ldots, m-1$ and $(a_1, \ldots, a_i)/\equiv \in C$, by the continuity of the multiplication in G, there exist open subsets

$$a \in V_{a_1a_2\ldots a_i},\ a_1 \in V_{a_1},\ a_2 \in V_{a_2},\ \ldots,\ a_i \in V_{a_i}$$

such that

$$aa_1\ldots a_i \in V_{a_1a_2\ldots a_i} V_{a_1} \ldots V_{a_i} \subset k(U).$$

Thus,

$$\bigcup_{a_1, \ldots, a_i \in C} V_{a_1}\ldots V_{a_i} \supset k(C).$$

By compactness of $k(C)$, then there exists a finite number of i-tuples, $(a_1^1, \ldots, a_i^1), \ldots, (a_1^n, \ldots, a_i^n)$ such that

$$\bigcup_{k=1}^{n} V_{a_1^k} V_{a_2^k} \ldots V_{a_i^k} \supset k(C).$$

Let $U = \bigcap_{k=1}^{n} V_{a_1^k a_2^k \ldots a_i^k}$ which is non-empty since it contains a. Then

$$\bigcup_{k=1}^{n} UV_{a_1^k} V_{a_2^k} \ldots V_{a_i^k} = U\left(\bigcup_{k=1}^{n} V_{a_1^k} V_{a_2^k} \ldots V_{a_i^k}\right) \subset k(U)$$

and hence

$$\bigcup_{k=1}^{n} U \times V_{a_1^k} \times \ldots \times V_{a_i^k} / \equiv \subset U.$$

This means,

$$L_U(C) \subset L_U\left(\bigcup_{k=1}^{n} V_{a_1^k} \times \ldots \times V_{a_i^k} / \equiv\right) \subset U,$$

in other words,

$$h(U) = L_U \subset (C, U).$$

Whence h is continuous and the final result follows.

Corollary 3.12: Let (A, [ ], t) be a topological m-group, F a compact subset of A, and U an open subset of A containing F. Then for each m-adic identity $(e_1, \ldots, e_{m-1})$ of A there exists open subsets $U_1, \ldots, U_{m-1}$ with $e_i \in U_i$, $i = 1, \ldots, m-1$ such that $[FU_1^{m-1}] \cup [U_1^{m-1}F] \subset U$. If A is locally compact, then $U_1, \ldots, U_{m-1}$ may be chosen so that $([FU_1^{m-1}] \cup [U_1^{m-1}F])^-$ is compact.

Proof. Let G be the containing group of A. By a result in Hewitt and Ross [8, 4.10] for topological groups, for F, U as above there exists an open set V containing the identity of G such that $FV \cup VF \subset U$. Then $e = e_1 \ldots e_{m-1} \in V$ and by the continuity of the operation in G there exist open subsets $U_1, \ldots, U_{m-1}$ with $e_i \in U_i$, $i = 1, \ldots, m-1$ such that $U_1 \ldots U_{m-1} \subset V$. Thus $[FU_1^{m-1}] \cup [U_1^{m-1}F] = (FU_1 \ldots U_{m-1}) \cup (U_1 \ldots U_{m-1}F) \subset FV \cup VF \subset U$.

If A is locally compact so that its containing group is locally compact, the second part of [8, 4.10] states

that V may be chosen so that $(FV \cup VF)^-$ is compact. Hence, $([FU_1^{m-1}] \cup [U_1^{m-1}F])^-$ as a closed subset of a compact space is compact.

Proposition 3.13: Let A_1 and A_2 be two m-groups such that A_1 is iseomorphic to A_2. Then their covering groups G_1 and G_2 are iseomorphic.

Proof. Let $f: A_1 \longrightarrow A_2$ be an iseomorphism and $G_1 = A_1 \cup \ldots \cup A_1^{m-1}$, $G_2 = A_2 \cup \ldots \cup A_2^{m-1}$.

Define $g: G_1 \longrightarrow G_2$ as follows. If $x \in G_1$, $x = x_1 \ldots x_i$ with $x_1, \ldots, x_i \in A_1$ let $g(x) = f(x_1)\ldots f(x_i)$. Then if $y \in G_1$ and $y = y_1 \ldots y_k$ with $y_1, \ldots, y_k \in A_1$, we have $g(xy) = g(x_1 \ldots x_i y_1 \ldots y_k) = f(x_1)\ldots f(x_i)f(y_1)\ldots f(y_k) = g(x)g(y)$ so g is a homomorphism. Next, suppose $g(x) = g(y)$. Then $f(x_1)\ldots f(x_i) = f(y_1)\ldots f(y_k)$ so $i = k$ and if $z_{i+1}, \ldots, z_m \in A_1$ we have $[f(x_1)\ldots f(x_i)f(z_{i+1})\ldots f(z_m)] = [f(y_1)\ldots f(y_i)f(z_{i+1})\ldots f(z_m)]$ and $f[x_1^i z_{i+1}^m] = f[y_1^i z_{i+1}^m]$. f is bijective, so $[x_1^i z_{i+1}^m] = [y_1^i z_{i+1}^m]$ which implies that $x_1 \ldots x_i = y_1 \ldots y_i$, i.e., $x = y$ and g is bijective.

Now let U be an open neighborhood of e_1 the identity of G_1. Then $e_1 = x_1 \ldots x_{m-1}$ with $(x_1, \ldots, x_{m-1})$ an m-adic identity, $x_i \in A_1$ $i = 1, \ldots, m-1$. U can be expressed as $U = x_1 \ldots x_{m-2}\, U'$ with U' open in A_1 and $x_{m-1} \in U'$. $g(U) = g(x_1 \ldots x_{m-2}\, U') = f(x_1)\ldots f(x_{m-2})f(U')$ and f(U') is open in A_2 so $f(x_1)\ldots f(x_{m-2})f(U') = g(U)$ is open in G_2 and g is an open function. In a similar manner it can be shown that g is continuous, so g is an iseomorphism.

Theorem 3.14: If a topological m-group A is compact, locally compact, σ-compact, or locally countably compact, then its containing group G is, respectively, each of these.

Proof. Since G is a topological group, multiplication is continuous, compactness, local compactness, and local countable compactness are clear. Since the finite union of a countable number of sets is countable and there are m-1 cosets of G, F compactness is clear.

Proposition 3.15: Let A be a topological m-group and let $A_1, A_2, \ldots, A_m$ be subsets of A. If A is T_o and $[x_{p(1)}^{p(m)}] = [x_1^m]$ for any permutation p and any choice of $x_i \in A_i$, then $[x_{p(1)}^{p(m)}] = [x_1^m]$ for any choice of $x_i \in \bar{A_i}$.

Proof. Let e be the identity of the containing group G of the m-group A. Let

$$H = \{(a_1, \ldots, a_m) \in G \times \ldots \times G \mid a_1 \ldots a_m a_{p(1)}^{-1} \ldots a_{p(m)}^{-1} = \{e\}\}.$$

Since A is T_o, G is T_o and $\{e\}$ is closed. H is the inverse image of a closed set under a continuous function and hence is closed in $G \times \ldots \times G$. Now A is closed as a subset of G, so $H \cap (A \times \ldots \times A)$ is closed. It is clear that $A_1 \times \ldots \times A_m \subset H$. Hence $(A_1 \times \ldots \times A_m)^- \subset \bar{A_1} \times \ldots \times \bar{A_m} \subset H$.

Proposition 3.16: If A is a T_o topological m-group and H is an abelian sub-m-semigroup or sub-m-group of A, then H is (respectively).

Proof. By Proposition 2.12 H^- is a sub-m-semigroup or sub-m-group if H is. By Proposition 3.15, H^- is abelian.

Lemma 3.17: Let A be a topological m-group with congruence relation R on it. If aR is compact for some $a \in R$, xR is compact for all $x \in A$.

Proof: Let $x \in A$ and choose $a_1, \ldots, a_{m-1} \in A$ such that $[a_1^{m-1}a] = x$. Then $[a_1^{m-1}(aR)] = xR$ is compact since by Theorem 2.6 it is the continuous image of a compact set.

Lemma 3.18: Let $A_1, \ldots A_m$ be a collection of sets such that $A_1, \ldots A_{m-1}$ are compact and A_m is closed. Then $[A_1 \ldots A_m]$ is closed.

Proof. Since in a group the product of compact sets is compact, we have $A_1 \ldots A_{m-1}$ compact in the containing group so $[A_1 \ldots A_m]$ is closed in A [8, 4.4].

Theorem 3.19: Let A be a topological m-group with a congruence relation R on A. If A/R and aR are compact for some $a \in A$, then A is compact.

Proof. By Proposition 2.23, since aR is compact, xR is compact for any $x \in A$. Let $e_1, \ldots, e_{m-1} \in A$ such that $(e_1, \ldots, e_{m-1})$ is an m-adic identity. As shown in Theorem 1.37, $N = e_1 \ldots e_{m-2}(e_{m-1}R)$ is a normal subgroup of the covering group G. Since G is a topological group, and $e_{m-1}R$ is compact, N is compact. We next show that G/N is compact. Since A/R is compact and $A/R = \{xN \mid x \in A\}$, $\{xN \mid x \in A\}$ is compact. Hence, in G, $y \cdot \{xN \mid x \in A\}$ is compact for any $y \in A$ and $y \cdot \{xN \mid x \in A\} = \{yxN \mid x \in A\} = \{zN \mid z \in A^2\}$. Continuing this process, we see that for any $i = 1, \ldots, m-1$, $\{xN \mid x \in A^i\}$ is compact so

$G/N = \bigcup_{i=1}^{m-1} \{xN \mid x \in A^i\}$ being a finite union of compact sets is compact. Thus N and G/N are compact, so G is compact [8, 5.25]. A being a closed subset of G is also compact.

Theorem 3.20: Let A be a locally compact, σ-compact topological m-group. Let f be a continuous homomorphism of A onto a locally countably compact T_o topological m-group A'. Then f is an open mapping.

Proof. Let G be the covering group of A and G' of A'. It is shown in the proof of Proposition 3.13 that f can be extended to a continuous open homomorphism f' between G and G'. Since A is locally compact and σ-compact, G also is. Since A' is locally countably compact and T_o, so is G'. Thus f' is an open mapping from G onto G' [8, 5.29]. Since A and A' are open subsets of G and G', respectively, the restriction of f' to A (which is f) is open.

CHAPTER IV

THE UNIVERSAL COVERING m-GROUP

In this chapter it will be shown that each arcwise connected, locally arcwise connected, and locally simply connected topological m-group with idempotent has an arcwise connected, locally arcwise connected and simply connected universal covering m-group [14, p. 232].

Definition 4.1: A regular, T_o, and second countable space A is arcwise connected (locally arcwise connected) if and only if for each pair a, b $\in$ A there exists a continuous function φ: [0,1] $\longrightarrow$ A such that $\varphi(0) = a$ and $\varphi(1) = b$ (for each a $\in$ A and every neighborhood U of a there exists a neighborhood V of a contained in U such that for all x $\in$ V there is a continuous function φ: [0,1] $\longrightarrow$ U such that $\varphi(0) = a$ and $\varphi(1) = x$).

Definition 4.2: A space A is simply connected (locally simply connected) if and only if for each a $\in$ A (for each neighborhood U of a there is a neighborhood V of a contained in U) such that for any continuous function φ: [0,1] $\longrightarrow$ A (φ: [0,1] $\longrightarrow$ V) such that $\varphi(0) = \varphi(1)$, then φ is homotopic to 0 in A (in U).

Lemma 4.3: If f: [0,1] $\longrightarrow$ A is homotopic to zero and φ: [0,1] $\longrightarrow$ A is an arbitrary continuous function,

then $\varphi * f$ is homotopic to φ, where $\varphi * f$ is defined as follows:

$$(\varphi * f)(t) = \begin{cases} \varphi(2t) & \text{for } 0 \leqq t \leqq \frac{1}{2} \\ f(2t-1) & \text{for } \frac{1}{2} \leqq t \leqq 1. \end{cases}$$

Proof: Note $f(0) = \varphi(1)$. Then the homotopy is effected by the following function:

$$F(s,t) = \begin{cases} \varphi\left(\frac{2t}{1+s}\right) & \text{for } 0 \leqq t \leqq \frac{1+s}{2} \\ f(0) & \text{for } \frac{1+s}{2} \leqq t \leqq 1. \end{cases}$$

Theorem 4.4: For each arcwise connected, locally arcwise connected, and locally simply connected topological m-group $(A, [\ldots], \tau)$ with an idempotent element e, there exists an arcwise connected, locally arcwise connected, and simply connected universal covering m-group $(\tilde{A}, [\ldots], \tilde{\tau})$ which is locally isomorphic to $(A, [\ldots], \tau)$ and such that if $\theta: \tilde{A} \longrightarrow A$ is the covering homomorphism, then $\tilde{A}/\theta \cdot \theta^{-1}$ is isomorphic to A.

Proof: Let $\tilde{A}$ be the family of homotopy classes of continuous functions $\varphi: [0,1] \longrightarrow A$ such that $\varphi(0) = e$. The homotopy class containing φ will be denoted by $\tilde{\varphi}$.

Consider any $\varphi \in \tilde{\varphi} \in \tilde{A}$ with $\varphi(1) = p \in A$. If $\mathfrak{U}$ is an open basis of the topology τ on A, then for each U containing p in $\mathfrak{U}$, let

$$U = \{\widetilde{\varphi * f} \mid f: [0,1] \longrightarrow U \text{ such that } f(0) = p\}$$

and $\tilde{\mathfrak{U}} = \{\tilde{U} \mid U \in \mathfrak{U}\}$. It is easy to see that if φ is replaced by any $\psi \in \tilde{\psi} \in \tilde{U}$, then U and ψ will determine exactly the same $\tilde{U}$ [14, p. 221].

(1) $\tilde{\mathfrak{U}}$ is an open basis for some topology $\tilde{\tau}$ on $\tilde{A}$.

If $\tilde{U}, \tilde{V} \in \tilde{\mathfrak{U}}$, so that $U, V \in \mathfrak{U}$, then $\tilde{U} \cap \tilde{V} \in \tilde{\mathfrak{U}}$. It is not difficult to show that $\widetilde{U \cap V} = \tilde{U} \cap \tilde{V}$.

(2) $(\tilde{A}, \tilde{\tau})$ is a T_o topological space.

Consider any pair $\tilde{\varphi}_1, \tilde{\varphi}_2 \in A$ such that $\tilde{\varphi}_1 \neq \tilde{\varphi}_2$. Let $\varphi_i \in \tilde{\varphi}_i$ $(i = 1, 2)$ such that $\varphi_i(1) = p_i$.

If $p_1 \neq p_2$, then since A is T_o there exists a neighborhood U of p_1 such that $p_2 \notin U$. In this case, $\tilde{\varphi}_1 \in \tilde{U}$ but $\tilde{\varphi}_2 \notin \tilde{U}$ and hence $\tilde{A}$ is also T_o.

If $p_1 = p_2$, then since A is locally simply connected, there is a neighborhood U of $p_1 = p_2$ such that every continuous $f: [0,1] \longrightarrow U$ with $f(0) = f(1) = p_1$ is homotopic to zero. If $\tilde{U} = \{\widetilde{\varphi_1 * f} \mid f: [0,1] \longrightarrow U \text{ with } f(0) = p_1\}$, then $\tilde{\varphi}_2 \notin \tilde{U}$; for, if $\tilde{\varphi}_2 \in \tilde{U}$, then there is an $f: [0,1] \longrightarrow U$ with $f(0) = p_1$ such that $\varphi_1 * f \in \tilde{\varphi}_2$, where

$$(\varphi_1 * f)(t) = \begin{cases} \varphi_1(2t) & \text{for } 0 \leqq t \leqq \frac{1}{2} \\ f(2t-1) & \text{for } \frac{1}{2} \leqq t \leqq 1. \end{cases}$$

Note: $\varphi_1(1) = f(0) = p_1 = (\varphi_1 * f)(1) = f(1)$. Thus f is homotopic to zero and $\tilde{\varphi}_1 = \widetilde{\varphi_1 * f} = \tilde{\varphi}_2$, which is a contradiction.

(3) If for $\varphi_i \in \tilde{\varphi}_i \in A$ $(i = i, \ldots, m)$ one defines $[\varphi_1 \varphi_2 \ldots \varphi_m](t) = [\varphi_1(t)\varphi_2(t)\ldots\varphi_m(t)]$ and $[\tilde{\varphi}_1 \tilde{\varphi}_2 \ldots \tilde{\varphi}_m] = [\widetilde{\varphi_1 \varphi_2 \ldots \varphi_m}]$, then $(\tilde{A}, [\ldots], \tilde{\tau})$ is a topological m-group.

Since $[\varphi_1 \varphi_2 \ldots \varphi_m](0) = [\varphi_1(0)\varphi_2(0)\ldots\varphi_m(0)] = [e^m] = e$ and $[\varphi_1 \varphi_2 \ldots \varphi_m]$ is a continuous function on $[0,1]$ to A, the above operation on A is clearly well defined. Associativity

follows from the following relations

$$\left[\left[\varphi_1^m\right]\varphi_{m+1}^{2m-1}\right](t) = \left[\left[\varphi_1^m\right](t)\varphi_{m+1}(t)\ldots\varphi_{2m-1}(t)\right]$$

$$= \left[\left[\varphi_1(t)\varphi_2(t)\ldots\varphi_m(t)\right]\varphi_{m+1}(t)\ldots\varphi_{2m-1}(t)\right]$$

$$= \left[\varphi_1(t)\ldots\varphi_{i-1}(t)\left[\varphi_i(t)\ldots\varphi_{i+m-1}(t)\right]\right.$$

$$\left.\varphi_{i+m}(t)\ldots\varphi_{2m-1}(t)\right] = \left[\varphi_1(t)\ldots\varphi_{i-1}(t)\right.$$

$$\left.\left[\varphi_i(t)\ldots\varphi_{i+m-1}(t)\right]\varphi_{i+m}(t)\ldots\varphi_{2m-1}(t)\right]$$

$$= \left[\varphi_1(t)\ldots\varphi_{i-1}(t)\left[\varphi_i^{i+m-1}\right](t)\varphi_{i+m}(t)\ldots\varphi_{2m-1}(t)\right]$$

$$= \left[\varphi_1^{i-1}\left[\varphi_i^{i+m-1}\right]\varphi_{i+m}^{2m-1}\right](t)$$

which holds for each $i = 2, \ldots, m$. For each $\varphi_i \in \tilde{\varphi}_i$ $(i = 1, \ldots, m-2)$, let $\varphi_{m-1}: [0,1] \longrightarrow A$ be the function such that $\varphi_{m-1}(t) = (\varphi_1(t), \ldots, \varphi_{m-2}(t))^{-1}$ for each $t \in [0,1]$. Since $[e^m] = e$, so that $(e, \ldots, e)$ is also an m-adic identity, then $\varphi_{m-1}(0) = e$. Define $(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{m-2})^{-1} = \tilde{\varphi}_{m-1}$. Then for each $\tilde{\varphi}$,

$$[\tilde{\varphi}_1\tilde{\varphi}_2\ldots\tilde{\varphi}_{m-1}\tilde{\varphi}] = \tilde{\varphi}.$$

Thus far, we have shown that $(\tilde{A}, [\ldots])$ is an m-group.

Next, we show that the functions

$$(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_m) \longrightarrow [\tilde{\varphi}_1\ldots\tilde{\varphi}_m]$$

$$(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{m-2}) \longrightarrow (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{m-2})^{-1}$$

are continuous. Let $\varphi_i \in \tilde{\varphi}_i \in \tilde{A}$ $(i = 1, \ldots, m)$ such that $\varphi_i(1) = p_i (i = 1, \ldots, m)$. Let $\tilde{V}$ be any neighborhood of $[\tilde{\varphi}_1\tilde{\varphi}_2\ldots\tilde{\varphi}_m] = \overbrace{[\varphi_1\varphi_2\ldots\varphi_m]}$ so that every element of $\tilde{V}$ is of the form $\overbrace{[\varphi_1\varphi_2\ldots\varphi_m]}*f$ for some $f: [0,1] \longrightarrow A$ such that

$f(0) = p = [p_1 p_2 \ldots p_m] = [\varphi_1(1)\varphi_2(1)\ldots\varphi_m(1)] = [\varphi_1^m](1) \in V.$

By the continuity of the m-ary operation on A, there exist neighborhoods U_i containing p_i $(i = 1, \ldots, m)$ such that

$$[U_1 U_2 \ldots U_m] \subset V.$$

Then $[\tilde{U}_1\tilde{U}_2\ldots\tilde{U}_m] \subset \tilde{V}$, for, if $\tilde{\psi}_i \in \tilde{U}_i$ $(i = 1, \ldots, m)$, then every $\psi_i \in \tilde{\psi}_i$ is of the form $\psi_i = \varphi_i * f_i$ $(i = 1, \ldots, m)$ for some $f_i\colon [0,1] \longrightarrow U_i$ such that $f_i(0) = p_i$. Thus,

$$\psi_i(t) = \begin{cases} \varphi_i(2t) & \text{for } 0 \leqq t \leqq \tfrac{1}{2} \\ f_i(2t-1) & \text{for } \tfrac{1}{2} \leqq t \leqq 1. \end{cases}$$

Then

$$[\psi_1(t)\psi_2(t)\ldots\psi_m(t)] = [\varphi_1(2t)\varphi_2(2t)\ldots\varphi_m(2t)]$$
$$= [\varphi_1^m](2t) \text{ for } 0 \leqq t \leqq \tfrac{1}{2} \text{ and } [\psi_1(t)\psi_2(t)\ldots\psi_m(t)]$$
$$= [f_1(2t-1)f_2(2t-1)\ldots f_m(2t-1)] = [f_1^m](2t-1) \text{ for}$$
$$\tfrac{1}{2} \leqq t \leqq 1.$$

Since $f_i(t) \in U_i$, then $[f_1^m]\colon [0,1] \longrightarrow V$. Also, if $[f_1^m](0)$ $= [f_1(0)f_2(0)\ldots f_m(0)] = [p_1 p_2 \ldots p_m] = p \in V$, then

$$[\tilde{\psi}_1\tilde{\psi}_2\ldots\tilde{\psi}_m] = [\overbrace{\psi_1\psi_2\ldots\psi_m}] = [\varphi_1^m] * [f_1^m] \in V.$$

Next consider $(\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_{m-2})^{-1} = \tilde{\varphi}_{m-1}$ where $\varphi_i(1) = p_i$ $(i = 1, \ldots, m-2)$ so that $(p_1, p_2, \ldots, p_{m-2})^{-1}$ $= p_{m-1}$. Let $\tilde{V}$ be a neighborhood of $\tilde{\varphi}_{m-1}$ with $\varphi_{m-1}(1)$ $= p_{m-1} \in V$. By the continuity of the inverse operation on A, there exist neighborhoods V_i of p_i $(i = 1, \ldots, m-2)$ such that

$$(V_1 \times V_2 \times \ldots \times V_{m-2})^{-1} \subset V.$$

Again, we claim

$$(\tilde{V}_1 \times \tilde{V}_2 \times \ldots \times \tilde{V}_{m-2})^{-1} \subset \tilde{V}.$$

For, if $(\tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_{m-2}) \in \tilde{V}_1 \times \tilde{V}_2 \times \ldots \times \tilde{V}_{m-2}$ and $\psi_i \in \tilde{\psi}_i$ $(i = 1, \ldots, m-2)$, then $\psi_i = \varphi_i * f_i (i = 1, \ldots, m-2)$ for some $f_i: [0,1] \longrightarrow V_i$. If $f_{m-1}: [0,1] \longrightarrow V$ is the continuous function such that $f_{m-1}(t) = (f_1(t), \ldots, f_{m-2}(t))^{-1}$, then $f_{m-1}(1) = (f_1(1), \ldots, f_{m-2}(1))^{-1} = p_{m-1} \in V$. Whence

$$(\tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_{m-2})^{-1} = \overbrace{(\psi_1, \psi_2, \ldots, \psi_{m-2})^{-1}}$$

$$= \overbrace{(\varphi_1, \varphi_2, \ldots, \varphi_{m-2})^{-1} * (f_1, f_2, \ldots f_{m-2})^{-1}}$$

$$= \overbrace{\varphi_{m-1} * f_{m-1}} \in \tilde{V}.$$

This completes the proof that $(\tilde{A}, [\ldots], \tilde{\tau})$ is a topological m-group.

(4) $(\tilde{A}, \tilde{\tau})$ is a second countable space.

If $|\mathfrak{U}| \leq \aleph_0$, then also $|\tilde{\mathfrak{U}}| \leq \aleph_0$.

(5) The covering function $\theta: \tilde{A} \longrightarrow A$ such that $\theta(\tilde{\varphi}) = \varphi(1) = p$ is an open continuous map which is locally a homeomorphism.

If U is an arbitrary neighborhood of p, then $\theta(\tilde{U}) \subset U$, obviously, so that θ is continuous.

Let $\tilde{\varphi} \in \tilde{A}$ and $\tilde{U}$ be a neighborhood of $\tilde{\varphi}$ defined by the neighborhood U of $\varphi(1) = p$. By local arcwise connectedness there exists a neighborhood V of p contained in U such that for any $x \in V$ there is a continuous function $f: [0,1] \longrightarrow U$ such that $f(0) = p$ and $f(1) = x$. This implies that $\varphi * f \in \tilde{U}$ and $\theta(\varphi * f) = (\varphi * f)(1) = f(2 \cdot 1 - 1) = f(1) = x$. Whence $V \subset \theta(\tilde{U})$, and θ is open.

Next, let $\tilde{\varphi} \in \tilde{A}$ and $\theta(\tilde{\varphi}) = \varphi(1) = p \in A$. Since A is locally simply connected, there is a neighborhood U of p such that every continuous $f: [0,1] \longrightarrow U$ such that $f(0) = f(1) = p$ is homotopic to zero in A. θ is one-to-one on $\tilde{U}$, for, suppose $\theta(\tilde{\psi}_1) = \theta(\tilde{\psi}_2)$ so that for some $f_i: [0,1] \longrightarrow U$ we have $f_i(0) = p (i = 1,2)$, and $\varphi * f_1 \in \tilde{\psi}_1$ and $\varphi * f_2 \in \tilde{\psi}_2$. If $f_1'(t) = f_1(1 - t)$ so that $f_2 * f_1': [0,1] \longrightarrow U$ is the continuous function such that

$$(f_2 * f_1')(t) = \begin{cases} f_2(2t) & \text{for } 0 \leqq t \leqq \frac{1}{2} \\ f_1'(2t-1) & \text{for } \frac{1}{2} \leqq t \leqq 1, \end{cases}$$

then $(f_2 * f_1')(0) = f_2(0) = e$ and $(f_2 * f_1')(1) = f_1'(1) = f_1(0) = e$. This means then that $f_2 * f_1'$ is homotopic to zero and therefore $\varphi * f_1$ is homotopic to $\varphi * f_2$ or $\tilde{\psi}_1 = \tilde{\psi}_2$.

Since θ is continuous, one-to-one, and open on $\tilde{U}$ it follows then that it is a homeomorphism on $\tilde{U}$.

By virtue of this, then

(6) $(\tilde{A}, \tilde{\tau})$ is also locally arcwise connected, locally simply connected, and regular.

(7) $(\tilde{A}, \tilde{\tau})$ is moreover arcwise connected.

Consider any $\tilde{\varphi} \in \tilde{A}$ and $\tilde{\omega} \in \tilde{A}$ where $\tilde{\omega}$ contains the null path. By homogenity of a topological m-group, it suffices to show that $\tilde{\omega}$ and $\tilde{\varphi}$ are connected by a continuous path. Let $\varphi \in \tilde{\varphi}$ and $\varphi_s: [0,1] \longrightarrow A$ be the continuous function such that $\varphi_s(t) = \varphi(st)$. For any fixed $s \in [0,1]$, note that $\varphi_s(0) = \varphi(0) = e$ and $\varphi_s(1) = \varphi(s) = p \in A$; and hence $\tilde{\varphi}_s \in \tilde{A}$. Define now $\Phi: [0,1] \longrightarrow A$ such that $\Phi(s) = \tilde{\varphi}_s$.

Then Φ is continuous [14, p. 223]. Moreover, $\Phi(0) = \tilde{\varphi}_0 = \tilde{\omega}$ and $\Phi(1) = \tilde{\varphi}_1 = \tilde{\varphi}$. Whence $\tilde{A}$ is arcwise connected.

(8) $(\tilde{A}, \tilde{\tau})$ is simply connected.

As in the previous, let $\tilde{\omega} \in \tilde{A}$ be the homotopy class containing the null path. Consider any continuous function Φ: $[0,1] \longrightarrow \tilde{A}$ such that $\Phi(0) = \tilde{\omega} = \Phi(1)$. Define φ such that $\varphi(t) = \theta(\Phi(t))$ where θ is the covering map. Define also for each $s \in [0,1]$, φ_s: $[0,1] \longrightarrow A$ such that $\varphi_s(t) = \varphi(st)$. Note that φ_s also depends continuously on t and $\varphi_s(0) = \varphi(0) = \theta(\Phi(0)) = \theta(\tilde{\omega}) = e$ so that $\tilde{\varphi}_s \in \tilde{A}$.

Firstly, observe that $\theta(\Phi(s)) = \varphi(s) = \theta(\tilde{\varphi}_s)$ for each $s \in [0,1]$.

We wish to show that $\tilde{\varphi}_s = \Phi(s)$ for all $s \in [0,1]$. For s = 0, the equality obviously holds:

$$\tilde{\varphi}_0 = \widetilde{\varphi(0)} = \tilde{\omega} = \Phi(0).$$

Let $\tilde{U}$ be a neighborhood of $\tilde{\varphi}_0 = \Phi(0)$ for which θ is a homeomorphism. Since both $\tilde{\varphi}_s$ and $\Phi(s)$ are continuous functions of s by an argument used in (7), then for some k sufficiently small we have

$$\tilde{\varphi}_{([0,k))} \subset \tilde{U} \text{ and } \Phi([0,k)) \subset \tilde{U}.$$

For each $x \in [0,k)$ so that $\tilde{\varphi}_x$, $\Phi(x) \in \tilde{U}$, then since $\theta(\tilde{\varphi}_x) = \theta(\Phi(x))$ and θ is one-to-one on U, then $\varphi_x = \Phi(x)$. This shows that $\varphi_s = \Phi(s)$ for all s less than k and hence by the continuity of the function φ_s and $\Phi(s)$ we obtain

$$\varphi_k = \lim_{s \longrightarrow k} \varphi_s = \lim_{s \longrightarrow k} \Phi(s) = \Phi(k).$$

Repeating the process now for k instead of 0, we should eventually show that the above relation holds for all $s \in [0,1]$. For $s = 1$, in particular, we have

$$\tilde{\varphi} = \tilde{\varphi}_1 = \Phi(1) = \Phi(0) = \tilde{\omega}$$

so that φ is homotopic to zero. Suppose that this homotopy is effected by the continuous function $F: [0,1] \times [0,1] \longrightarrow A$ so that $F(0,t) = \varphi(t)$, $F(1,t) = \varphi(0) = e$, $F(s,0) = \varphi(0) = e$, $F(s,1) = \varphi(1) = \varphi(0) = e$.

For each fixed s and t, define

$$G(s,t): [0,1] \longrightarrow A$$

to be the function such that $G(s,t)(x) = F(s, tx)$. Since $F(s, tx)$ is continuous in s, t, and x, then $G(s,t)$ is also continuous in s and t. For each fixed s and t, note also that

$$G(s,t)(0) = F(s,0) = \varphi(0) = e$$

$$G(s,t)(1) = F(s,t) = p.$$

Thus for each s and t, $\widetilde{G(s,t)} \in \tilde{A}$ and $\widetilde{G(s,t)}$ also depends continuously on s and t. The following relations now show that $\widetilde{G(s,t)}$ effects the homotopy of Φ and 0:

$$\widetilde{G(0,t)}(x) = \widetilde{F(0,tx)} = \widetilde{\varphi_t}(x) \text{ or } \widetilde{G(0,t)} = \tilde{\varphi}_t = \Phi(t)$$

$$\widetilde{G(1,t)}(x) = \widetilde{F(1,tx)} = \widetilde{\varphi(0)} = \tilde{\omega} \text{ or } \widetilde{G(1,t)} = \tilde{\omega}$$

$$\widetilde{G(s,0)}(x) = \widetilde{F(s,0)} = \widetilde{\varphi(0)} = \tilde{\omega} \text{ or } \widetilde{G(s,0)} = \tilde{\omega} = \Phi(0)$$

$$\widetilde{G(s,1)}(x) = \widetilde{F(s,x)} = \tilde{p} = \tilde{\omega} = \Phi(1).$$

(9) θ is a homomorphism of $\tilde{A}$ onto A.

It is clearly onto. Let $\varphi_i \in \tilde{\varphi}_i \in \tilde{A}$ such that $\varphi_i(1) = p_i (i = 1, 2, \ldots, m)$. Then

$$\theta([\varphi_1\varphi_2\cdots\varphi_m]) = \theta(\widetilde{[\varphi_1\varphi_2\cdots\varphi_m]}) = [\varphi_1\varphi_2\cdots\varphi_m](1)$$
$$= [\varphi_1(1)\varphi_2(1)\cdots\varphi_m(1)] = [p_1p_2\cdots p_m] =$$
$$= [\theta(\varphi_1)\theta(\varphi_2)\cdots\theta(\varphi_m)].$$

The following theorem proves that $\tilde{A}$ is the universal covering m-group, i.e., $\tilde{A}$ is unique up to iseomorphism.

Theorem 4.5: Let $h:A' \longrightarrow A$ be a continuous, open, local homeomorphism, homomorphism of an arcwise connected, locally arcwise connected and simply connected topological m-group $(A', [\,], U)$ with idempotent e' such that $h(e') = e$ onto the given topological m-group $(A, [\,], t)$. Then A' and $\tilde{A}$ are iseomorphic.

Proof. Let $\theta:\tilde{A} \longrightarrow A$ be as previously defined and define a map $k:A' \longrightarrow \tilde{A}$ as follows: Let $\varphi' \in A'$ and let $\varphi:[0,1] \longrightarrow A'$ such that $\varphi(0) = e'$, $\varphi(1) = \varphi'$ with φ a continuous function. Then $h \circ \varphi$ defines a curve in A with $(h \circ \varphi)(0) = e$ and $(h \circ \varphi)(1) = h(\varphi')$. Since A is simply connected, the choice of curve $\varphi(t)$ connecting e' and φ' is unique up to homotopy, and since h is continuous, the image under h of any two such choices will be homotopic. The function $h \circ \varphi:[0,1] \longrightarrow A$ and hence defines a unique point $\widetilde{h \circ \varphi}$ in $\tilde{A}$. Define $k(\varphi') = \widetilde{h \circ \varphi}$. It is clear that k is well defined.

Now let U' be an open subset of A' and let $\varphi_o' \in U'$. Let $U = \{\varphi \mid \varphi:[0,1] \longrightarrow U', \varphi(0) = \varphi_o'$ and each φ is continuous$\}$ and let ψ be continuous with $\psi:[0,1] \longrightarrow A$ such that $\psi(0) = e'$, $\psi(1) = \varphi_o'$. Since U' is open in A', h(U') is

open in A and A being locally simply connected implies that there exists an open set V in A about the point $h(\varphi_o')$, $V \subset h(U')$ such that any closed curve in V is null homotopic, or that any two curves contained in V beginning at φ_o' and ending at the same point are homotopic.

Letting $W = \{h \circ \varphi | \varphi \in U, h \circ \varphi:[0,1] \longrightarrow V$ such that $(h \circ \varphi)(0) = h(\varphi_o')\}$, we see that

(1) $W \subset h \circ U = \{h \circ \varphi | \varphi \in U\}$,

(2) $\psi * (h \circ \varphi) | h \circ \varphi \in W\}$ defines an open set in $\tilde{A}$, precisely the open set $\tilde{V}$, and

(3) $\tilde{V} \subset k(U')$.

Now, (1) is clear and (2) is clear if we note that the choice of V gives only one function (up to homotopy) connecting $h(\varphi_o')$ and any point in V and this function will be homotopic to the h-image of the corresponding function in A'. $\tilde{V} \subset k(U')$ by the simple connectedness of A' and the definition of k. Hence, k is an open function. Next we must show that k is continuous. Let $\varphi_o' \in A'$ and let $\tilde{V}$ be a basic open set in $\tilde{A}$ about $k(\varphi_o')$. Since θ is an open map, $\theta(\tilde{V})$ is open in A and A is locally simply connected so there exists a simply connected open set W in A such that $W \subset \theta(\tilde{V})$ and $\theta(k(\varphi_o')) \in W$. Then $h^{-1}(W)$ is an open set in A' about φ_o' such that $k(h^{-1}(W)) \subset \tilde{V}$. To see this, let $\psi:[0,1] \longrightarrow A'$ such that $\psi(0) = e'$, $\psi(1) = \varphi_o'$ and ψ is continuous. Let $X = \{\varphi | \varphi:[0,1] \longrightarrow h^{-1}(W)$ $\varphi(0) = \psi(1)$ and each φ is continuous$\}$. Since A' is simply connected,

$\psi * X = \{\psi * \varphi | \varphi:[0,1] \longrightarrow h^{-1}(W),\ \varphi(0) = \psi(1)$ and each φ is continuous$\}$ represents all continuous functions $\psi:[0,1] \longrightarrow A'$ such that $f(0) = e'$ and $f(1) \in H^{-1}(W)$. Since W is simply connected, $h \circ X = \{h \circ \varphi | \varphi:[0,1] \longrightarrow h^{-1}(W)\ \varphi(o) = \psi(1)$ and each φ is continuous$\}$ represents all continuous function $g:[0,1] \longrightarrow W$ such that $g(0) = (h \circ \varphi)(0) = h(\varphi_o')$. Hence, $k(h^{-1}(W)) = \overbrace{h \circ (\psi * X)} = \overbrace{(h \circ \psi) * (h \circ X)} = \tilde{W} \subset \tilde{V}$.

Next, let ψ', $\varphi' \in A'$ and suppose $k(\psi') = k(\varphi')$. If $k(\psi') = k(\varphi')$, the functions $\psi:[0,1] \longrightarrow A'$ and $\varphi:[0,1] \longrightarrow A'$ must be homotopic and hence, $\psi(1) = \varphi(1)$, i.e., $\psi' = \varphi'$. Hence, k is 1-1.

Now let $\varphi_1', \ldots, \varphi_m' \in A'$ with associated functions $\varphi_i:[0,1] \longrightarrow A'$ such that $\varphi_i(1) = \varphi_i'$, $i = 1, 2, \ldots, m$. Then $k[\varphi_1' \ldots \varphi_m'] = \overbrace{h \circ [\varphi_1 \ldots \varphi_m]} = \overbrace{[h \circ \varphi_1 \ldots h \circ \varphi_m]}$ (since h is a homomorphism) $= [\overbrace{h \circ \varphi_1} \ldots \overbrace{h \circ \varphi_m}] = [k(\varphi_1') \ldots k(\varphi_m')]$. Thus, $k:A' \longrightarrow \tilde{A}$ is an iseomorphism.

Recalling that the iseomorphism of $\tilde{A}/\theta \cdot \theta^{-1}$ to A was demonstrated in Theorem 2.29, we see that the truth of Theorem 4.4 has been demonstrated.

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BIOGRAPHICAL SKETCH

Robert Lee Richardson was born February 8, 1937, in Burlington, Vermont. In 1954, he graduated from Northfield High School and went to Castleton State College, graduating in 1958. From 1958 to 1960, he taught at Middlebury Union High School leaving in 1960 to attend the University of Notre Dame. In 1961, he received his Master of Science degree with major in mathematics from the University of Notre Dame and became an instructor at Norwich University. In 1962, he came to the University of Florida as a half-time interim instructor to do further graduate work, remaining until August, 1966 when he received his Doctor of Philosophy degree.

Robert Lee Richardson is married to the former Eleanor Rita Dundon of Orwell, Vermont. He is the father of three children, Robert Lee, Jr., Mary Margaret, and Patrick Joseph.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August, 1966

E. Russell Jones
Dean, College of Arts and Sciences

Dean, Graduate School

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